The Yamabe problem on stratified spaces.

Ilaria Mondello

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COMMUNAUTÉ D'UNIVERSITÉS ET ÉTABLISSEMENTS

UNIVERSITÉ DE NANTES

Thèse de Doctorat

Ilaria MONDELLO

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Le Problème de Yamabe sur les espaces stratifiés.

JURY

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UNIVERSITÉ DE NANTES
FACULTÉ DES SCIENCES ET TECHNIQUES

ÉCOLE DOCTORALE SCIENCES ET TECHNOLOGIES DE L'INFORMATION ET DES MATHÉMATIQUES

The Yamabe Problem on Stratified Spaces

Thèse de Doctorat de l'Université de Nantes

Spécialité : Mathématiques et Applications

Ilaria Mondello

Directeur de Thèse : Gilles Carron

Mais une des innombrables particularités qui distinguent l'homme de la bestiole, c'est qu'il en veut plus. Et même quand il a la quantité suffisante, c'est la qualité qu'il réclame. Les faits bruts ne lui suffisent pas, il lui faut aussi les « pourquoi », les « comment » et les « jusqu'où ».

Daniel Pennac

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Dans l'imaginaire collectif, faire une thèse en mathématiques est un travail d'ermite, un effort solitaire face à un grand problème abstrait, tandis que la vie continue son chemin loin d'un bureau faiblement éclairé... Heureusement, la réalité est bien différente, et ces trois dernières années ont été très intenses, accompagnées par le soutien indispensable de nombreuses personnes, mathématiciens et non. Toutes les remercier va prendre du temps et de la place dans cette thèse, mais ce temps et cette place ont été mérités.

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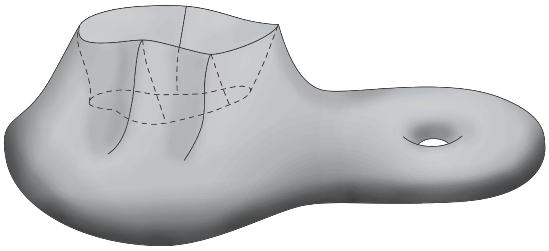
Introduction

L'étude des variétés différentiables munies d'une métrique Riemannienne constitue un domaine prospère des mathématiques depuis le dix-neuvième siècle, et elle a connu d'innombrables avancements tout au long du vingtième, grâce à ses interactions fructueuses avec la topologie et l'analyse. Néanmoins, certains phénomènes très naturels incitent à élargir les horizons des géomètres : par exemple, il est un résultat bien connu et élémentaire que le quotient d'une variété Riemannienne par un groupe n'est pas forcément une variété Riemannienne. De plus, si on considère une suite de variétés compactes lisses M_i, munies d'une métrique riemannienne g_i, sa limite de Gromov-Hausdorff, quand elle existe, n'est pas en général une variété. Il y a alors plusieurs approches possibles : on peut par exemple introduire des variétés avec singularités, comme les orbifolds ; on peut également étudier des espaces métriques, plutôt que des variété lisses, et chercher à étendre des outils de géométrie Riemannienne, notamment la courbure, à ces espaces. Cette direction à été entreprise par exemple par A. Alexandrov, qui a introduit une notion de courbure basée sur la comparaison des triangles, et plus récemment par les travaux de D. Bakry et M. Émery et ceux de J. Lott, C. Villani, et K.-T. Sturm, dans le contexte du transport optimal de mesures.

Cette thèse s'occupe d'étudier une classe particulière d'espaces métriques singuliers, les espaces stratifiés. Ces derniers sont apparus dans plusieurs branches des mathématiques. Ils ont été introduits d'abord par H. Whitney [Whi47], avec l'idée de partitionner un espace topologique en sous-éléments plus simples, des "complexes de variétés", bien agencés entre eux. La dénomination de stratification est due à R. Thom [Tho69], qui s'est servi des espace stratifiés pour l'étude de la stabilité des applications lisses entre variétés. De plus, les espaces stratifiés fournissent un cadre approprié à la reformulation de la dualité de Poincaré pour l'homologie d'intersection sur des espaces singuliers ([GM88], [Pfl01]). D'un point de vue analytique, J. Cheeger [Che83] a inauguré dans les années 80 l'étude de l'analyse spectrale pour des variétés avec singularités coniques ou à coins, qui sont des cas particuliers d'espaces stratifiés ; plusieurs contributions ont été apportées par R. Melrose et par son école à travers l'outil des opérateurs pseudo-différentiels, et par une école allemande qui comprend K.-T. Sturm, J. Brüning, K. Schulze, W. Ballmann.

Il est aussi très naturel d'étudier les espaces stratifiés avec des outils de géométrie différentielle, car ils peuvent être construits très facilement à partir de variétés compactes lisses. Par exemple, considérons une sphère $\mathbb{S}^2$ et une rotation d'angle $2\pi/n$ pour un entier n. Le quotient de la sphère par le groupe engendré par cette rotation a la forme d'un ballon de rugby, avec deux singularités coniques aux points fixes de la rotation.

Or, une surface à singularités coniques est l'exemple le plus simple possible d'espace stratifié. En effet, les espaces stratifiés généralisent la notion de singularité conique isolée, dans le sens suivant : un espace stratifié X est un espace métrique compact qui peut être décomposé en un lieu régulier, c'est-à-dire une variété ouverte, lisse de dimension n et dense dans X, et un lieu singulier. Ce dernier est l'union disjointe de plusieurs composantes connexes, les *strates singulières*, de dimensions différentes. Un point dans une strate singulière possède un voisinage qui est homéomorphe au produit entre un boule euclidienne et un cône sur une base, qui s'appelle *link*, ou étoile, de la strate. Le link peut être à priori un espace stratifié. Nous renvoyons au Chapitre 1 pour une définition détaillée. Pour se faire une idée, on peut imaginer une variété de dimension 3, et considérer des singularités coniques le long d'un courbe, comme sur la figure suivante :



Le voisinage d'un point appartenant à la courbe singulière est alors le produit d'un intervalle dans $\mathbb{R}$ avec un cône qui a comme base un cercle $\mathbb{S}^1$. Ce genre de construction peut être généralisé à des dimensions différentes, en donnant lieu à des variétés avec *simple edges*, qui sont des espaces stratifiés dont les links sont des variétés compactes lisses. Dans le Chapitre 1 nous donnons également plusieurs exemples dans lesquels le link peut être un espace singulier.

On peut aussi définir une métrique g sur un espace stratifié : c'est une métrique riemannienne lisse sur le lieu régulier ; près d'une strate singulière, la métrique doit rendre compte de la structure locale de l'espace, et elle a la forme d'une métrique produit entre celle euclidienne et une métrique conique. Par conséquent, un espace stratifié est aussi muni d'une mesure dv_g, et on peut définir des objets analytiques, en suivant [ACM14] : notamment les espaces L^p et de Sobolev, qui seront de première importance dans la suite. Pour un espace stratifié X muni d'un métrique g, l'espace de Sobolev $W^{1,2}(X)$ est l'adhérence de l'ensemble des fonctions Lipschitz sur X pour la norme usuelle :

$$\|u\|_{1,2}^2 = \int_X |u|^2 dv_g + \int_X |du|^2 dv_g.$$

Il est alors possible de montrer que l'inégalité de Sobolev :

$$\|u\|_{\frac{2n}{n-2}} \leq C_s \|u\|_{1,2}$$

est vérifiée pour toute fonction u dans $W^{1,2}(X)$, et que les injections de Sobolev de $W^{1,2}(X)$ dans $L^p(X)$, pour p compris entre 1 et $2n/(n-2)$, ont les mêmes propriétés que dans le cas des variétés compactes. De plus l'opérateur Laplacien Δ_g peut être défini, en tant qu'extension de Friedrichs de la forme quadratique appropriée.

Le but principal de cette thèse consiste à montrer comment certains résultats classiques de géométrie Riemannienne et d'analyse globale sur les variétés compactes peuvent être étendus aux espaces stratifiés, et d'étudier ensuite les conséquences de ces résultats sur la résolution d'un problème d'analyse géométrique, le problème de Yamabe. Celui-ci consiste à chercher une métrique à courbure scalaire constante parmi la classe conforme d'une métrique donnée g, c'est-à-dire parmi l'ensemble des métriques qui sont obtenues par multiplication de g par une fonction lisse et positive :

$$[g] = \{\tilde{g} = fg, \text{ pour } f \in C^\infty(M), f > 0\}.$$

Dans le cas d'une surface de Riemann compacte, il existe toujours une métrique conforme avec courbure de Gauss constante égale à $1, 0$ ou -1, d'après le théorème d'uniformisation de Poincaré. De plus, cela implique que la surface de départ est conforme à un quotient respectivement de la sphère $\mathbb{S}^2$, du plan euclidien $\mathbb{R}^2$ ou de l'espace hyperbolique $\mathbb{H}^2$. Pour une variété compacte lisse de dimension supérieure ou égale à 3, le problème a été formulé par H. Yamabe [Yam60] en 1960, puis résolu à travers les contributions de plusieurs mathématiciens, N. Trudinger, T. Aubin, R. Schoen ([Tru68], [Aub76a], [Sch84]). L'approche utilisée repose sur la compréhension d'un invariant conforme, la *constante de Yamabe*, qui est la borne inférieure de l'intégrale de la courbure scalaire parmi les métriques conformes de volume unitaire :

$$Y(M,[g]) = \inf\left\{\int_M \mathrm{Scal}_{\tilde{g}} dv_{\tilde{g}}, \text{ t.q. } \tilde{g} \in [g] \text{ et } \mathrm{Vol}_{\tilde{g}}(M) = 1\right\}.$$

Une métrique qui atteint cette constante s'appelle *métrique de Yamabe* et est à courbure scalaire constante, mais réciproquement, une métrique à courbure scalaire constante n'est pas nécessairement une métrique de Yamabe, car elle n'est pas nécessairement minimisante.

Les questions au départ de ce travail proviennent du récent article de K. Akutagawa, G. Carron et R. Mazzeo [ACM14], qui fournit le cadre approprié pour étudier le problème de Yamabe sur les espaces stratifiés, par une approche similaire à celle entreprise par N. Trudinger et T. Aubin dans le cas de variétés compactes lisses. Les auteurs démontrent, entre autres, qu'une métrique de Yamabe existe si la courbure scalaire satisfait une condition d'intégrabilité et si la constante de Yamabe de l'espace est strictement inférieure à un invariant conforme, la *constante de Yamabe locale*. Or, la valeur de cet invariant n'est connue qu'en très peu de cas et trouver des conditions pour que l'inégalité soit stricte n'est pas un problème trivial. Il serait aussi souhaitable d'obtenir un résultat de rigidité dans le cas d'égalité.

Comme il advient souvent dans la recherche, et peut-être pas seulement dans la recherche, les meilleures questions ne sont pas celles qui mènent à une réponse, mais plutôt celles qui en engendrent d'autres. Ainsi le problème initial consistant à calculer la constante de Yamabe locale m'a amenée à étudier les propriétés d'espaces stratifiés dont le tenseur de Ricci est minoré sur le lieu régulier. Pour une variété compacte lisse avec tenseur de Ricci minoré, nous avons un résultat bien connu, dû à A. Lichnerowicz et à M. Obata :

Théorème (Lichnerowicz-Obata). *Soit (M,g) une variété Riemannienne compacte lisse de dimension n, telle que le tenseur de Ricci est minoré par une constante positive k, $Ric_g \geq kg$. Alors la première valeur propre non nulle du Laplacien est supérieure ou égale à nk, avec égalité si et seulement si (M,g) est isométrique à la sphère $\mathbb{S}^n$ de rayon $1/\sqrt{k}$ munie de la métrique canonique.*

De plus, toute variété complète dont le tenseur de Ricci est minoré est compacte :

Théorème (Myers). *Soit (M^n,g) une variété Riemannienne lisse de dimension n, telle que le tenseur de Ricci est minoré par une constante positive k. Alors le diamètre de (M^n,g) est inférieur ou égal à $\pi/\sqrt{k}$.*

Nous pouvons étendre une partie de ces résultats à une large classe d'espaces stratifiés, que l'on peut considérer comme des espaces à tenseur de Ricci minoré par une constante positive. La première condition à imposer est, évidemment, que le tenseur de Ricci soit minoré sur le lieu régulier. On doit de plus imposer une condition qui concerne la strate singulière de codimension égale à deux. En effet, le modèle local autour d'un point de cette strate est le produit d'une boule euclidienne avec un cône sur un cercle : l'angle de ce cône peut être inférieur ou supérieur à 2π, selon que le rayon du cercle soit inférieur ou supérieur à 1. Cela donne lieu à deux situations géométriques différentes, car un cône est un espace métrique de courbure positive au sens d'Alexandrov si son angle est inférieur à 2π , et à courbure négative si son angle est supérieur à 2π. Donc la présence d'une strate de codimension deux dont le link est un cercle de rayon supérieur à 1 introduit en un certain sens de la courbure négative, qui crée un obstacle pour prouver des résultats analogues aux précédents.

Nous considérons alors la notion d'espace stratifié *admissible*, qui en plus d'avoir un tenseur de Ricci minoré sur le lieu régulier, par la dimension de l'espace moins un, ne possède pas de strate singulière de codimension deux et angle du cône strictement supérieur à 2π. Dans ce cadre nous pouvons prouver le théorème suivant :

Théorème A (Lichnerowicz singulier). *Soit (X,g) un espace stratifié admissible de dimension n. Alors la première valeur propre du Laplacien est supérieure ou égale à n.*

K. Bacher et K.-T. Sturm [BS14] ont prouvé un théorème "à la Lichnerowicz" pour des cônes et des suspensions sphériques dont la base est une variété Riemannienne compacte à tenseur de Ricci minoré par une constante positive, c'est-à-dire des espaces stratifiés qui ont respectivement une ou deux singularités coniques isolées. Les techniques qu'ils utilisent se basent sur l'inégalité de courbure-dimension de Sturm-Lott-Villani. Notre résultat s'applique pour des singularités plus générales que les singularités coniques isolées, et sa preuve est plus proche de la démonstration classique dans le cas régulier, car elle se fonde principalement sur la formule de Bochner-Lichnerowicz.

Notre version singulière du théorème de Lichnerowicz affirme, en termes analytiques, l'existence d'un trou spectral. Celui-ci, couplé avec l'existence d'une inégalité de Sobolev sur un espace stratifié, nous permet d'obtenir une inégalité de Sobolev améliorée, pour laquelle les constantes qui apparaissent ne dépendent que de la dimension et du volume de l'espace :

Théorème B. *Soit (X,g) un espace stratifié admissible de dimension n. Pour tout p compris entre 1 et $2n/(n-2)$ et tout f dans l'espace de Sobolev $W^{1,2}(X)$, l'inégalité de Sobolev suivante est vérifiée :*

$$V^{1-\frac{2}{p}}\, \|f\|_p^2 \leq \|f\|_2^2 + \frac{p-2}{n}\, \|df\|_2^2 \,,$$

où V est le volume de X pour la métrique g.

Ce résultat est inspiré d'un théorème de D. Bakry, contenu dans [Bak94]. Les travaux de ce dernier avec M. Ledoux ont également montré que le théorème de Myers peut être démontré en utilisant uniquement des outils analytiques, pourvu que le trou spectral existe et qu'une inégalité de Sobolev analogue à celle prouvée dans le Théorème B soit vérifiée. Il s'ensuit que nous pouvons appliquer la démonstration de [BL96] et en déduire :

Corollaire C (Myers singulier)**.** *Soit (X,g) un espace stratifié admissible de dimension n. Alors son diamètre est inférieur ou égal à π.*

En outre, nous démontrons que la borne π pour le diamètre est atteinte si et seulement si la première valeur propre non-nulle du Laplacien est égale à la dimension de l'espace. Plus précisément, nous obtenons le résultat suivant :

Corollaire D. *Soit (X,g) un espace stratifié admissible de dimension n. Les trois affirmations suivantes sont équivalentes :*

(i) *La première valeur propre non-nulle du Laplacien Δ_g est égale à n.*

(ii) *Le diamètre de X est égal à π.*

(iii) *Il existe des fonctions extrémales pour l'inégalité de Sobolev.*

Par fonctions extrémales, nous entendons fonctions qui atteignent l'égalité dans l'inégalité de Sobolev du Théorème B.

Il serait également intéressant d'obtenir un résultat de rigidité dans l'esprit de celui de M. Obata, lorsque la première valeur propre non-nulle du Laplacien est égale à la dimension de l'espace. Le fait que $\lambda_1(\Delta_g) = n$ soit équivalent à avoir un diamètre égal à π rend très vraisemblable ce genre de résultat, mais, en mettant en place une preuve inspirée par celle classique, on se heurte à des obstacles liés à l'unicité et à la régularité des géodésiques minimisantes : le problème n'est pas trivial à résoudre.

Il existe des versions similaires des théorèmes "à la Lichnerowicz-Obata" et "à la Myers" pour des espaces métriques mesurés qui satisfont une inégalité de courbure-dimension : leurs preuves reposent sur un récent théorème de splitting dû à N. Gigli [Gig13] (voir [Ket14]). Nos résultats sont moins généraux, mais leurs démonstrations ont l'avantage d'utiliser principalement sur des outils de géométrie riemannienne classique.

Les Théorèmes A et B ont des conséquences directes sur l'étude du problème de Yamabe pour des espaces stratifiés. En effet, le deuxième théorème nous permet de déduire une borne inférieure sur la constante de Yamabe, qui est atteinte dans le cas d'une métrique d'Einstein :

Corollaire E. *Soit (X,g) un espace stratifié admissible de dimension n. Alors sa constante de Yamabe satisfait :*

$$Y(X,[g]) \geq \frac{n(n-2)}{4}\operatorname{Vol}_g(X)^{\frac{2}{n}} = \left(\frac{\operatorname{Vol}_g(X)}{\operatorname{Vol}(\mathbb{S}^n)}\right)^{\frac{2}{n}} Y_n,$$

où Y_n est la constante de Yamabe de la sphère canonique de dimension n. L'égalité est atteinte dans l'inégalité précédente si la métrique est une métrique d'Einstein.

Ceci étend un théorème analogue de J. Petean dans [Pet09], qui affirme l'existence de la même minoration pour la constante de Yamabe d'un cône sur une variété riemannienne compacte avec tenseur de Ricci minoré par une constante positive. Ce résultat découle d'une étude des domaines isopérimétriques des cônes, similaire à celle faite par F. Morgan et M. Ritoré dans [MR02]. Il serait intéressant d'étendre cette approche aux cônes sur les espaces stratifiés admissibles, dont la dimension d'Hausdorff du lieu singulier est inférieure à $(n-3)$.

L'un des intérêts du résultat précédent est qu'il peut être appliqué pour calculer la constante de Yamabe locale d'un espace stratifié dont les links soient munis d'une métrique d'Einstein. Cette condition est motivée par le fait que, si la courbure de chaque link Z de dimension d est égale à $d(d-1)$, alors la courbure de tout l'espace satisfait la condition d'intégrabilité nécessaire pour appliquer le théorème d'existence de [ACM14]. Nous obtenons :

Théorème F. *Soit (X,g) un espace stratifié de dimension n avec strates singulières X^j et links (Z_j,k_j) de dimension d_j. Supposons qu'aucun des links n'est un cercle de rayon supérieur à 1, et que pour tout j la métrique k_j sur le link Z_j est une métrique d'Einstein. Alors la constante de Yamabe locale de (X,g) est égale à :*

$$Y_\ell(X) = \inf_j \left\{ \left(\frac{\operatorname{Vol}_{k_j}(Z_j)}{\operatorname{Vol}(\mathbb{S}^{d_j})}\right)^{\frac{2}{n}} Y_n \right\},$$

où Y_n est la constante de Yamabe de la sphère $\mathbb{S}^n$.

Ce résultat permet de calculer la constante de Yamabe locale dans de nombreux cas, comme par exemple un espace stratifié avec une strate de codimension égale à deux, lorsque l'angle du cône est inférieur à 2π. Il s'applique en particulier aux orbifolds, et il étend un théorème de K. Akutagawa et B. Botvinnik, valable pour les orbifolds à singularités isolées.

Le cas qui n'est pas inclus dans le théorème précédent est celui d'une strate de codimension deux pour laquelle l'angle est supérieur à 2π. Nous traitons cette situation en utilisant une approche différente, qui se base sur une relation entre une inégalité de Sobolev optimale et l'inégalité isopérimétrique, et que nous expliquons à présent. En suivant une stratégie basée sur le lissage de la métrique conique et sur l'étude de profils isopérimétriques, nous démontrons le résultat suivant :

Théorème G. *Considérons le produit* $\mathbb{R}^{n-2} \times C(\mathbb{S}^1_a)$*, où* $\mathbb{S}^1_a$ *est le cercle de rayon* $a > 1$*, muni de la métrique* g *produit entre la métrique euclidienne* ξ *sur* $\mathbb{R}^{n-2}$ *et la métrique conique* $dr^2 + (ar)^2 d\theta^2$ *sur* $C(\mathbb{S}^1_a)$ *:*

$$g = \xi + dr^2 + (ar)^2 d\theta^2.$$

Soit $I_g :]0, \mathrm{vol}_g(X)/2] \to \mathbb{R}$ *le profil isopérimétrique associé à* g *:*

$$I_g(v) = \inf\{\mathrm{Vol}_g(\partial E), E \subset X, \text{ t.q. } \partial E \text{ est lisse par morceaux }, \mathrm{Vol}_g(E) = v\}.$$

Alors I_g *coïncide avec le profil isopérimétrique euclidien* $I_n(v) = c_n v^{1-\frac{1}{n}}$*.*

La preuve montre que le cône $C(\mathbb{S}^1_a)$ peut être approché par des surfaces de Cartan-Hadamard, qui sont complètes, simplement connexes et à courbure sectionnelle minorée ; elle utilise un résultat d'A. Weil [Wei26], qui a prouvé la conjecture de Cartan-Hadamard en dimension deux, et un résultat d'A. Ros sur le profil isopérimétrique d'un produit riemannien.

Cela nous permet d'adapter un argument classique de G. Talenti [Tal76] (et prouvé indépendamment par T. Aubin dans [Aub76b]) sur l'inégalité de Sobolev optimale et d'en déduire :

Corollaire H. *Soit* (X^n, g) *un espace stratifié avec une seule strate de codimension deux et angle du cône supérieur à* 2π*. Alors sa constante de Yamabe locale coïncide avec celle de la sphère canonique de dimension* n*.*

Nous pouvons donc améliorer le Théorème F en enlevant l'hypothèse sur la strate de codimension deux et le reformuler de la façon suivante :

Théorème I. *Soit* (X, g) *un espace stratifié de dimension* n *avec strates singulières* X^j *et links* (Z_j, k_j) *de dimension* d_j*. Si pour tout* j *la métrique* k_j *sur le link* Z_j *est une métrique d'Einstein, la constante de Yamabe locale de* (X, g) *est égale à :*

$$Y_\ell(X) = \inf_j \left\{ Y_n, \left(\frac{\mathrm{Vol}_{k_j}(Z_j)}{\mathrm{Vol}(\mathbb{S}^{d_j})} \right)^{\frac{2}{n}} Y_n \right\}.$$

Le résultat précédent répond donc à la question de calculer la constante de Yamabe locale pour tout espace stratifié dont les links soient munis d'une métrique d'Einstein. Nous présentons dans la suite une autre approche possible pour traiter ce problème, inspirée par un autre résultat de M. Obata. Soit (M^n, g) une variété compacte lisse de dimension supérieure ou égale à 3 : si g est une métrique d'Einstein, alors elle atteint la constante de Yamabe, et toute autre métrique conforme à g avec courbure scalaire constante est homothétique à g.

La démonstration de ce résultat consiste à prouver que s'il existe une métrique conforme à g, à courbure scalaire constante, et qui n'est pas homothétique à g, alors il existe une fonction propre relative à la valeur propre n, et (M^n, g) est isométrique à la sphère canonique. Nous montrons un résultat analogue en adaptant un argument utilisé par J. Viaclovsky dans [Via10] :

Théorème J. *Soit (X^n, g) un espace stratifié admissible d'Einstein. S'il existe une métrique conforme à g, non homothétique à g et avec courbure scalaire constante, alors la métrique d'Einstein g est une métrique de Yamabe.*

La difficulté de cette approche est qu'elle nécessite un résultat d'existence d'une métrique de Yamabe sur l'espace stratifié. Nous pouvons toutefois donner une classe d'exemples pour laquelle il y a existence. Si on considère une variété compacte d'Einstein (Z^d, k) nous avons les équivalences conformes suivantes :

$$\begin{aligned}\left(\mathbb{H}^{n-d} \times Z^d, [g_{\mathbb{H}} + k]\right) &\cong \left(\mathbb{R}^{n-d-1} \times Z^d, [\xi + dr^2 + r^2 k]\right) \\ &\cong \left(C(\hat{Z}), [dt^2 + \sin^2(t) h]\right).\end{aligned}$$

où $\hat{Z}$ et sa métrique h sont définis par :

$$\begin{aligned}\hat{Z} &= \left[0, \frac{\pi}{2}\right] \times \mathbb{S}^{n-d-3} \times Z^d \\ h &= d\psi^2 + \cos^2(\psi) g_{\mathbb{S}} + \sin^2(t) k.\end{aligned}$$

Le cône sur $\hat{S}$ est un espace stratifié compact de dimension n, muni de la métrique d'Einstein $dt^2 + \sin^2(t)h$.

Or, un résultat dû à K. Akutagawa, prouvé par N. Grosse dans [Gro13], concernant l'existence d'une métrique de Yamabe sur des variétés ouvertes et complètes, s'applique au produit $\mathbb{H}^{n-d} \times Z^d$. Nous donnons une preuve alternative et plus directe, utilisant essentiellement la technique d'itération de Moser, de ce théorème de N. Grosse :

Théorème. *Soit (M^n, g) un variété complète lisse de dimension $n \geq 3$. Supposons qu'il existe un sous-groupe d'isométries $\Gamma \subset Isom(M)$ et un compact K tels que pour tout x dans M il existe une isométrie γ de Γ qui envoie x dans K. Si la courbure scalaire de g est strictement positive et si la constante de Yamabe $Y(M, [g])$ est strictement inférieure à Y_n, alors il existe une métrique de Yamabe dans la classe conforme de g.*

Comme conséquence, nous obtenons que le produit $\mathbb{H}^{n-d} \times Z^d$ admet une métrique de Yamabe tant que sa courbure scalaire est strictement positive. Si nous supposons donc que la dimension d de Z est strictement supérieure à $n/2$, il existe donc une métrique de Yamabe sur le cône $C(\hat{Z})$. Nous pouvons alors appliquer le Théorème J et en déduire que la métrique d'Einstein atteint la constante de Yamabe. Ceci permet aussi de retrouver la valeur de la constante de Yamabe locale du Théorème F pour un espace stratifié de dimension n, dont les links sont des variétés compactes, lisses, d'Einstein, avec dimension supérieure à $n/2$.

Nous concluons cette introduction par quelques perspectives du travail à venir, qui motivent les résultats présentés ci-dessus. Connaître la valeur de la constante de Yamabe locale ouvre naturellement la voie à plusieurs questions : sous quelles conditions peut-on avoir inégalité stricte, et donc existence d'une métrique de Yamabe, entre constante de Yamabe locale et globale ? Que se passe-t-il si on a égalité entre les deux ?

Dans le cas d'une variété compacte lisse (M^n, g), la constante de Yamabe locale est égale à celle de la sphère Y_n, et une métrique de Yamabe existe lorsque la constante de Yamabe est strictement inférieure à Y_n. Or, T. Aubin a montré que pour toute variété de dimension supérieure ou égale à 6 dont la métrique n'est pas localement conformément plate, l'inégalité stricte

$$Y(M^n, [g]) < Y_n,$$

est vérifiée. Son argument est local et utilise des fonctions test appropriées, dont le support est contenu dans une boule.

Si nous considérons un espace stratifié de dimension supérieure ou égale à 6, muni d'une strate singulière de codimension deux, dont l'angle est supérieur à 2π, nous savons grâce aux Corollaire H que sa constante de Yamabe locale est égale à Y_n. Si sa métrique n'est pas localement conformément plate, en utilisant les mêmes fonctions test que T. Aubin autour d'un point du lieu régulier, nous récupérons alors le même résultat, c'est-à-dire la constante de Yamabe *globale* de l'espace stratifié est strictement inférieure à sa constante de Yamabe *locale* Y_n. En particulier, dans ce cas, une métrique de Yamabe existe.

Le problème est différent en dimension petite, égale à 3, 4 ou 5 et pour une métrique localement conformément plate sur le lieu régulier. R. Schoen [Sch84] a montré, en se basant sur ses travaux avec S.T. Yau à propos du théorème de la masse positive, que si $Y(M^n, [g])$ coïncide avec Y_n, alors la variété est conforme à la sphère canonique $\mathbb{S}^n$. Un théorème de rigidité analogue a été démontré par E. Witten en toute dimension lorsque la variété est supposée spin. Il est donc raisonnable de croire que certains des outils développés par R. Schoen et E. Witten peuvent s'étendre au cas des espaces stratifiés avec une strate de codimension deux et angle du cône supérieur à 2π.

Si on considère un espace stratifié de dimension n avec une seule strate de codimension deux et angle α du cône inférieur à 2π, nous avons montré que sa constante de Yamabe locale est strictement inférieure à celle de la sphère. En particulier, elle est égale à :

$$Y_\ell(X) = \left(\frac{\alpha}{2\pi}\right)^{\frac{2}{n}} Y_n.$$

Par conséquent, utiliser des fonctions test à support dans le lieu régulier comme dans le cas précédent ne donne aucune information sur la relation entre constante de Yamabe locale et globale. On pourrait s'inspirer de l'étude proposée par J.M. Lee and T.H. Parker du développement de la fonction de Green associée au Laplacien conforme (voir [LP87]) : en adaptant cette approche au voisinage d'un point singulier, on chercherait à trouver une relation entre les coefficients de ce développement et l'angle α. Cela pourrait suggérer les conditions à imposer pour que l'on ait inégalité stricte entre constante de Yamabe locale et globale.

Enfin, il faut également rappeler que J. Viaclovsky à exhibé dans [Via10] un exemple d'orbifold avec une singularité conique dont la constante de Yamabe coïncide avec celle locale, et pour lequel il n'existe pas de métrique de Yamabe. Il est donc possible que d'autres exemples de non-existence se présentent, et il serait intéressant de les étudier (ou classifier ?).

Plan de la thèse

Le Chapitre 1 est dédié aux définitions géométriques et analytiques concernant les espaces stratifiés, aux exemples, et à la présentation de plusieurs résultats connus à propos des espaces de Sobolev et des équations de Schrödinger dans le cadre singulier. Il se base principalement sur [Klo09], [ALMP12], [ACM14] and [ACM15]. Nous prouvons également un résultat de régularité pour le gradient d'une solution de l'équation de Schrödinger, qui se fonde sur la géométrie spectrale des links, et qui représente un outil technique nécessaire à plusieurs preuves dans la suite.

Dans le Chapitre 2 nous collectons et démontrons les nouveaux résultats à propos des espaces stratifiés admissibles : on y trouve en particulier les démonstrations des Théorèmes A, B et du Corollaire D.

Le Chapitre 3 présente brièvement le problème de Yamabe sur les variétés compactes lisses et les résultats obtenus dans [ACM14] dans le cadre des espaces stratifiés. Cette partie ne contient pas de résultat nouveau, mais elle est importante pour motiver et comprendre la suite.

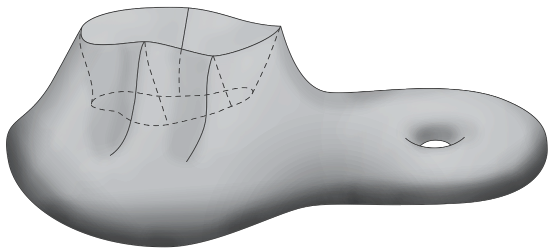
Enfin, le dernier Chapitre est dédié à tisser les liens entre les résultats prouvés pour les espaces stratifiés admissibles et la question initiale consistant à calculer la constante de Yamabe locale. Nous y démontrons le Corollaire E et le Théorème F. Une deuxième partie du chapitre présente le Théorème G avec son Corollaire H. Nous concluons par l'approche "à la Obata" avec la preuve du Théorème J et la reformulation du théorème de N. Grosse pour les variétés presque homogènes.

Introduction

The study of differentiable manifolds endowed with a Riemannian metric constitutes a prosperous domain in mathematics since the nineteenth century, and it had countless developments all along the twentieth, thanks to its successful interactions with both topology and analysis. Nevertheless, some very natural phenomena push geometers to extend their horizons: for example, it is a well known and elementary fact that the quotient of a Riamannian manifold by a group is not necessarily a Riemannian manifold. Moreover, if we consider a sequence of compact smooth manifolds M_i, endowed with a Riemannian metric g_i, its Gromov-Hausdorff limit, when it exists, is not a manifold in general. There are then various possible approaches: one can for example introduce manifolds with singularities, like orbifolds; it is also possible to study metric spaces, instead of compact smooth manifolds, and to try and extend some tools of Riemannian geometry, like curvature, to these spaces. This direction has been taken for example by A. Alexandrov, who introduced a notion of curvature based on the comparison of triangles, and more recently by the works of D. Bakry and M. Émery and those of J. Lott, C. Villani and K.-T. Sturm, in the context of optimal transport of measures.

This thesis is dedicated to the study of a particular class of singular metric spaces, stratified spaces. These latter has appeared in various domains of mathematics. First, they have been introduced by H. Whitney [Whi47], with the idea of partitioning a topological space in simpler elements, "complexes of manifolds", which must be well glued together. The denomination of stratification is due to R. Thom [Tho69], who used stratified spaces to study the stability of smooth applications between manifolds. Moreover, stratified spaces furnish an appropriate context to reformulate Poincaré's duality for the intersection homology of singular spaces ([GM88], [Pfl01]). From an analytical point of view, during the eighties J. Cheeger launched the study of spectral analysis on manifolds with conical singularities or simple edges, which are particular cases of stratified spaces. Many contributions have been provided by R. Melrose and his school through the tool of pseudo-differential operators and micro-local analysis, and by a German school including K.-T. Sturm, J. Brüning, K. Schulze, W. Ballmann.

It also very natural to study stratified spaces with tools coming from differential geometry, since they can be constructed very easily starting from compact smooth manifolds. For example, let us consider a sphere $\mathbb{S}^2$ and a rotation of angle $2\pi/n$ for an integer n. The quotient of the sphere by the group generated by this rotation has the form of an American football, with two conical singularities at the two fixed points of the rotation. Now, a surface with conical singularities is the simplest possible example of stratified space. Indeed, stratified spaces generalize the notion of isolated conical singularity, in the following sense: a stratified space X is a compact metric space which can be decomposed in a regular set, that is an open smooth manifold of dimension n, dense in X, and a singular set. This latter is the disjoint union of different connected components, the *singular strata*, of different dimensions. A point in a singular stratum possesses a neighbourhood which is homeomorphic to the product between an Euclidean ball and a cone over a base, called *link* of the stratum. The link may be a priori a stratified space. We refer to Chapter 1 for a detailed definition. In order to get an idea, one can imagine a manifold of dimension 3, and consider conical singularities along a curve, like in the figure below:



The neighbourhood of a point belonging to the singular curve is then be the product between an interval in $\mathbb{R}$ and a cone which has a circle $\mathbb{S}^1$ as a basis. This kind of construction can be generalized to different dimensions, leading to manifolds with simple edges: these are stratified spaces whose links are compact smooth manifolds. In the first chapter we give various examples that explain why the link can be a singular space as well.

It is also possible to define a metric g on a stratified space: this is be a smooth Riemannian metric on the regular set; near to a singular stratum, it must respect the local structure of the space, and it has the form of a product metric between the Euclidean one and a conic metric. As a consequence, a stratified space is also endowed of a measure dv_g and one can define some analytical objects, following [ACM14]: in particular the L^p and Sobolev spaces, which are going to be fundamental in what follows. For a stratified space X endowed with a metric g, the Sobolev space $W^{1,2}(X)$ is the closure of the set of Lipschitz functions on X with the usual norm:

$$\|u\|_{1,2}^2 = \int_X |u|^2 dv_g + \int_X |du|^2 dv_g.$$

It is then possible to show that the Sobolev inequality

$$\|u\|_{\frac{2n}{n-2}} \leq C_s \|u\|_{1,2}$$

holds for any function u in $W^{1,2}(X)$, and that the Sobolev embeddings of $W^{1,2}(X)$ in $L^p(X)$ for p between 1 and $2n/(n-2)$, have the same properties as in the case of compact smooth manifolds. Furthermore, the Laplacian operator Δ_g can be defined: it is the Friedrichs extension of the appropriate quadratic form.

The main goal of this thesis consists in showing how certain classical results in Riemannian geometry and global analysis on compact smooth manifolds can be extended to stratified spaces, and to study then their consequences on the resolution of a problem in geometric analysis, the Yamabe problem. This latter consists in looking for a metric with constant scalar curvature in the conformal class of a given metric g, that is the set of all the metrics which are obtained from g by multiplication by a smooth positive function:

$$[g] = \{\tilde{g} = fg, \text{ pour } f \in C^\infty(M), f > 0\}.$$

In the case of a compact Riemannian surface, there always exists a conformal metric with constant Gauss curvature equal to 1, 0 or -1, thanks to Poincaré's uniformization theorem. Moreover, this implies that the given surface is conformal to a quotient, respectively of the sphere $\mathbb{S}^2$, of the Euclidean plane $\mathbb{R}^2$ or of the hyperbolic plane $\mathbb{H}^2$. For a compact smooth manifold of dimension larger than three, the problem has been formulated by H. Yamabe [Yam60] in 1960 and solved afterwards through the works of various mathematicians, N. Trudinger, T. Aubin, R. Schoen ([Tru68], [Aub76a], [Sch84]). The approach which has been used relies on the comprehension of a conformal invariant, the *Yamabe constant*, which is the infimum of the integral of the scalar curvature among the conformal metrics of volume one:

$$Y(M, [g]) = \inf \left\{ \int_M \mathrm{Scal}_{\tilde{g}} dv_{\tilde{g}}, \text{ t.q. } \tilde{g} \in [g] \text{ et } \mathrm{Vol}_{\tilde{g}}(M) = 1 \right\}.$$

A metric which attains this constant is called a *Yamabe metric* and it has constant scalar curvature, but vice versa, a metric with constant scalar curvature is not necessarily a Yamabe metric, since it is not necessarily minimizing.

The starting questions of this work come from the recent article by K. Akutagawa, G. Carron and R. Mazzeo [ACM14], which gives the appropriate setting to study the Yamabe problem on stratified spaces, through an approach similar to the one developed by N. Trudinger and T. Aubin in the case of compact smooth manifolds. Moreover, the authors prove, among others, that a Yamabe metric exists if the scalar curvature satisfies an integrability condition and if the Yamabe constant of the space is strictly less than a conformal invariant, the *local Yamabe constant*. Now, the explicit value of this invariant is known only in a few cases, and to find conditions for which the inequality is strict is not trivial. It would also be desirable to have a rigidity result in the case of equality.

As it often happens in research, and maybe not only in research, the best questions are not the ones leading to an answer, but the ones that give birth to other questions. The initial problem of computing the local Yamabe constant led me to study the properties of stratified spaces whose Ricci tensor is bounded by below on the regular set. For a compact smooth manifold with Ricci tensor bounded by below we have a well-known result, due to A. Lichnerowicz and M. Obata:

Theorem (Lichnerowicz-Obata). *Let (M,g) be a compact Riemannian manifold of dimension n, with Ricci tensor bounded by below by a positive constant k, $Ric_g \geq kg$. Then the first non-zero eigenvalue of the Laplacian is larger or equal to kn, with equality if and only if (M,g) is isometric to the sphere $\mathbb{S}^n$ of radius $1/\sqrt{k}$ endowed with the canonical metric.*

Moreover, any complete manifold with Ricci tensor bounded by below is in fact compact:

Theorem (Myers). *Let (M,g) be a complete Riemannian manifold of dimension n, with Ricci tensor bounded by below by a positive constant k. Then the diameter of (M^n,g) is less or equal than $\pi/\sqrt{k}$.*

We can extend part of these results to a large class of stratified spaces, which we can considered as spaces with Ricci tensor bounded by below by a positive constant. The first condition to ask is clearly that the Ricci tensor is bounded by below on the regular set. We have to ask for a further condition concerning the singular stratum of codimension equal to two. Indeed, the local model around a point of this stratum is the product between an Euclidean ball and a cone over a circle: the angle of this cone can be smaller or larger than 2π, depending on whether the radius of the circle is smaller or larger than 1. This leads to two different geometric situations, since a cone is a metric space of positive curvature in the sense of Alexandrov if its angle is smaller than 2π, and of negative curvature if its angle is larger than 2π. Therefore, the presence of a stratum of codimension two and cone angle larger than 2π introduces in some sense negative curvature, and it generates an obstacle to prove results analogous to the previous ones.

We consider then the notion of *admissible* stratified space, which in addition to having Ricci tensor bounded by below on the regular set by the dimension of the space minus one, does not posses any singular stratum of codimension two and cone angle larger than 2π. In this context we prove the following theorem:

Theorem A (Singular Lichnerowicz). *Let (X,g) be an admissible stratified space of dimension n. Then the first non-zero eigenvalue of the Laplacian operator is larger than or equal to n.*

K. Bacher and K.-T. Sturm [BS14] have proven a Lichneorowicz theorem for cones and spherical suspension whose basis is a compact Riemannian manifold with Ricci tensor bounded by below, that is stratified space with respectively one and two isolated conical singularities. The techniques that they use are based on a curvature-dimension condition in the sense of Sturm-Lott-Villani. Our result applies to more general singularities than conical isolated singularities, and its proof is more similar to the classical demonstration for the regular case, since it is mainly based on the Bochner-Lichnerowicz formula.

Our singular version of the Lichnerowicz theorem affirms, in analytical terms, the existence of a spectral gap. This latter, together with the existence of a Sobolev inequality on a stratified space, allows us to obtain a refined Sobolev inequality, for which the constants appearing in the right-hand side only depends on the volume and on the dimension of the space:

Theorem B. *Let (X, g) be an admissible stratified space of dimension n. For any p between 1 and $2n/(n-2)$, and for any function f in $W^{1,2}(X)$, the following Sobolev inequality holds:*

$$V^{1-\frac{2}{p}} \|f\|_p^2 \le \|f\|_2^2 + \frac{p-2}{n} \|df\|_2^2 ,$$

where V is the volume of X with respect to the metric g.

This result is inspired by a theorem due to D. Bakry, contained in [Bak94]. The works of this latter and M. Ledoux also showed that the Myers theorem can be proven by using analytical tools only, provided that the spectral gap exists and a Sobolev inequality analogous to the one proven in Theorem B holds. As a consequence, we can apply the proof of [BL96] and deduce:

Corollary C (Singular Myers). *Let (X, g) be an admissible stratified space of dimension n. Then its diameter is smaller or equal than π.*

Furthermoree, we also prove that the diameter attains its upper bound π if and only if the first non-zero eigenvalue of the Laplacian is equal to the dimension of the space. More precisely, we have the following result:

Corollary D. *Let (X, g) be an admissible stratified space of dimension n. The following statements are equivalent:*

(i) *The first non-zero eigenvalue of the Laplacian Δ_g is equal to n.*

(ii) *The diameter of X is equal to π.*

(iii) *There exist extremal functions for the Sobolev inequality.*

With extremal functions we mean functions attaining the equality in the Sobolev inequality obtained in Theorem B.

It would be also interesting to get a rigidity result in the spirit of Obata's theorem, when the first non-zero eigenvalue of the Laplacian is equal to the dimension of the space. The fact that $\lambda_1(\Delta_g) = n$ is equivalent to having diameter equal to π makes this kind of result very probable, but, when we try to adapt the classical proof to this case, we encounter some obstacles related to the uniqueness and the regularity of minimizing geodesics: the problem is not trivial to solve.

There exist similar versions of the Lichnerowicz-Obata and Myers theorems for metric-measure spaces satisfying a curvature-dimension condition, which rely on a recent splitting theorem by N. Gigli [Gig13] (see [Ket14]). Our results are less general, but our proofs have the advantage to mainly use tools of classical Riemannian geometry.

Theorems A and B have direct consequences on the study of the Yamabe problem for stratified spaces. Indeed, the first theorem allows us to deduce a lower bound on the Yamabe constant, which is attained in the case of Einstein metric:

Corollary E. *Let (X, g) be an admissible stratified space of dimension n. Then its Yamabe constant satisfies*

$$Y(X,[g]) \geq \frac{n(n-2)}{4} \operatorname{Vol}_g(X)^{\frac{2}{n}} = \left(\frac{\operatorname{Vol}_g(X)}{\operatorname{Vol}(\mathbb{S}^n)}\right)^{\frac{2}{n}} Y_n.$$

where Y_n is the Yamabe constant of the sphere $\mathbb{S}^n$. The equality is attained in the previous inequality if the metric is an Einstein metric.

This extends an analogous theorem by J. Petean in [Pet09], which states the existence of the same bound by below for the Yamabe constant of a cone over a compact Riemannian manifold with Ricci tensor bounded by below by a positive constant. This results relies on the study of the isoperimetric domains in cones, similar to the one of F. Morgan and M. Ritoré in [MR02]. It would be interesting to extend this argument to cones over admissible stratified spaces, provided that the Hausdorff dimension of the singular set is smaller than $(n-3)$.

One of the interests of the previous result is that it can be applied to compute the local Yamabe constant of a stratified space whose links are endowed with an Einstein metric. This condition is motivated by the fact that, if the scalar curvature of each link Z of dimension d is equal to $d(d-1)$, then the curvature of the whole space satisfies the integrability condition which is necessary to apply the existence theorem in [ACM14]. We obtain:

Theorem F. *Let (X, g) be a stratified space of dimension n with singular strata X^j and links (Z_j, k_j) of dimension d_j. Assume that any of the links is a circle of radius larger than one, and that for every j the metric k_j on Z_j is an Einstein metric. Then the local Yamabe constant of (X, g) is equal to:*

$$Y_\ell(X) = \inf_j \left\{ \left(\frac{\operatorname{Vol}_{k_j}(Z_j)}{\operatorname{Vol}(\mathbb{S}^{d_j})}\right)^{\frac{2}{n}} Y_n \right\}$$

This result allows us to compute the local Yamabe constant of numerous cases, like for example a stratified space with a stratum of codimension equal to two and cone angle less than 2π. In particular it can be applied to orbifolds, and it extends a theorem of K. Akutagawa and B. Botvinnik holding for orbifolds with isolated singularities.

The case which is not included in the previous theorem is the one of a stratum of codimension two and cone angle larger than 2π. We are going to treat this situation with a different approach, which depends on the relation between an optimal Sobolev inequality and the isoperimetric inequality, and which we explain in what follows. By following a strategy based on smoothing the conic metric and on studying the isoperimetric profiles, we prove the following theorem:

Theorem G. *Let us consider the product $\mathbb{R}^{n-2} \times C(\mathbb{S}^1_a)$, where $\mathbb{S}^1_a$ is the circle of radius $a > 1$ endowed with the metric g, product between the euclidean metric ξ on $\mathbb{R}^{n-2}$ and the conic metric $dr^2 + (ar)^2 d\theta^2$ on $C(\mathbb{S}^1_a)$:*

$$g = \xi + dr^2 + (ar)^2 d\theta^2.$$

Let $I_g : (0, \mathrm{vol}_g(X)/2] \to \mathbb{R}$ *the isoperimetric profile associated to g:*

$$I_g(v) = \inf\{\mathrm{Vol}_g(\partial E), E \subset X, \text{ s.t. } \partial E \text{ is piecewise smooth }, \mathrm{Vol}_g(E) = v\}.$$

Then I_g *coincides with the euclidean isoperimetric profile* $I_n(v) = c_n v^{1-\frac{1}{n}}$.

The proof shows that the cone $C(\mathbb{S}^1_a)$ can be approximated by Cartan-Hadamard surfaces, which are complete, simply connected, with negative sectional curvature. It uses a result of A. Weil [Wei26], which has proven the Cartan-Hadamard conjecture in dimension 2, and a result of A. Ros about the isoperimetric profile of a Riemannian product.

This allows us to adapt a classical argument due to G. Talenti [Tal76] (and proven indipendently by T. Aubin in [Aub76b]) about the optimal Sobolev inequality in order to deduce:

Corollary H. *Let* (X, g) *be a stratified space of dimension* n *with one singular stratum of codimension two and cone angle larger than* 2π*. Then its local Yamabe constant coincides with the one of the sphere* $\mathbb{S}^n$*.*

We can then improve Theorem F by taking away the hypothesis about the stratum of codimension two, and we can reformulate it in the following way:

Theorem I. *Let* (X, g) *be a stratified space of dimension* n *with singular strata* X^j *and links* (Z_j, k_j) *of dimension* d_j*. If for every* j *the metric* k_j *on the link* Z_j *is an Einstein metric, then the local Yamabe constant of* (X, g) *is equal to:*

$$Y_\ell(X) = \inf_j \left\{ Y_n, \left(\frac{\mathrm{Vol}_{k_j}(Z_j)}{\mathrm{Vol}(\mathbb{S}^{d_j})} \right)^{\frac{2}{n}} Y_n \right\}$$

The previous result answers completely to the question of computing the local Yamabe constant of any stratified space whose links are endowed of an Einstein metric. We present here another possible approach to treat this problem, which is inspired by another result by M. Obata. Let (M^n, g) be a compact smooth manifold of dimension larger or equal than three: if g is an Einstein metric, then it attains the Yamabe constant, and any other metric conformal to g with constant scalar curvature is homothetic to g.

The proof consists in showing that if there exists another metric conformal to g with constant scalar curvature, which is not homothetic to g, then there exists an eigenfunction for the eigenvalue n, and (M^n, g) is isometric to the sphere. We show an analogous result by adapting an argument used by J. Viaclovsky in [Via10]:

Theorem J. *Let* (X^n, g) *be an admissible Einstein stratified space. If there exists a conformal metric, not homothetic to* g *and with constant scalar curvature, then the Einstein metric is a Yamabe metric.*

The difficulty of this approach is that it needs a result of existence of a Yamabe metric on a stratified space. Nevertheless, we can give a class of examples for which the existence holds. If we consider a compact Einstein manifold (Z^d, k) we have the following conformal equivalences:

$$\begin{aligned}\left(\mathbb{H}^{n-d} \times Z^d, [g_{\mathbb{H}} + k]\right) &\cong \left(\mathbb{R}^{n-d-1} \times Z^d, [\xi + dr^2 + r^2 k]\right) \\ &\cong \left(C(\hat{Z}), [dt^2 + \sin^2(t)h]\right).\end{aligned}$$

where $\hat{Z}$ and its metric h are defined by:

$$\begin{aligned}\hat{Z} &= \left[0, \frac{\pi}{2}\right] \times \mathbb{S}^{n-d-3} \times Z^d \\ h &= d\psi^2 + \cos^2(\psi) g_{\mathbb{S}} + \sin^2(t)k.\end{aligned}$$

The cone over $\hat{S}$ is a compact stratified space of dimension n, endowed with the Einstein metric $dt^2+\sin^2(t)h$. Now, a theorem due to K. Akutagawa, proven by N. Grosse [Gro13], which concerns the existence of a Yamabe metric on complete manifolds, can be applied to the product $\mathbb{H}^{n-d} \times Z^d$. We give an alternative and more direct proof, which uses the Moser iteration technique, of this result of N. Grosse:

Theorem. *Let (M^n, g) be a complete smooth manifold of dimension $n \geq 3$. Assume that there exists a subgroup og isometries $\Gamma \subset \mathrm{Isom}(M)$ and a compact K such that for any $x \in M$ there exists an isometry γ in Γ which sends x in K. If the scalar curvature of g is positive and if the Yamabe constant $Y(M, [g])$ is strictly smaller than Y_n, then there exists a Yamabe metric in the conformal class of g.*

As a consequence, we have that the product $\mathbb{H}^{n-d} \times Z^d$ admits a Yamabe metric provided that it has positive scalar curvature: then if we assume that d is strictly larger than $n/2$, the exists a Yamabe metric on the cone $C(\hat{Z})$. Therefore we can apply Theorem J and deduce from it that the Einstein metric attains the Yamabe constant. This also allows one to find the value of the local Yamabe constant of Theorem F for a stratified space of dimension n, whose links are compact smooth Einstein manifolds of dimension strictly larger than $n/2$.

We conclude this introduction with some perspectives of the work to come, which motivate the results presented above. Knowing the value of the local Yamabe constant naturally opens the way to further questions: under which conditions can we have the strict inequality between the local and global Yamabe constants, and thus existence of a Yamabe metric? What happens when the equality occurs?

In the case of a compact smooth manifold (M^n, g), the local Yamabe constant is equal to the one of the sphere Y_n, and a Yamabe metric exists when the Yamabe constant is strictly smaller than Y_n. Now, T. Aubin showed that for any compact manifold of dimension larger than or equal to 6 and not locally conformally flat metric, the strict inequality:

$$Y(M^n, [g]) < Y_n,$$

is satisfied. His argument is local and it is based on using the appropriate test functions, whose support is contained in a ball. If we consider a stratified space of dimension larger than or equal to 6, with only one singular stratum of codimension two and cone angle larger than 2π, we know thanks to Corollary H that its local Yamabe constant is equal to Y_n. If the metric is not locally conformally flat, by using the same test functions as T. Aubin around a regular point, we then get the same result, that is, the *global* Yamabe constant of the space is strictly smaller than its *local* Yamabe constant Y_n. In particular, a Yamabe metric exists in this case.

The problem is different for small dimension, equal to 3, 4 or 5, and for a locally conformally flat metric on the regular set. R. Schoen [Sch84] has shown, thanks to his works with S.T. Yau concerning the positive mass theorem, that if $Y(M^n, [g])$ coincides with Y_n, then the manifold is conformally equivalent to the canonical sphere $\mathbb{S}^n$. An analogous rigidity theorem has been proven by E. Witten in any dimension when the manifold is supposed to be spin. It is then reasonable to believe that some of the tools developed by R. Schoen and E. Witten can be extended to the case of stratified spaces with one stratum of codimension two and cone angle larger than 2π.

If we consider a stratified space with one stratum of codimension two and cone angle α smaller than 2π, we have shown that its local Yamabe constant is strictly smaller than the one of the sphere. In particular, it is equal to:

$$Y_\ell(X) = \left(\frac{\alpha}{2\pi}\right)^{\frac{2}{n}} Y_n.$$

As a consequence, to use test functions with support in a ball of the regular set as in the previous case does not give any information about the relationship between the global and local Yamabe constant. One could take inspiration from the study proposed by J.M. Lee and T.H. Parker of the expansion of the Green function associated to the conformal Laplacian (see [LP87]): by adapting this approach in a neighbourhood of a singular point one could seek for a relationship between the coefficients of this expansion and the cone angle α. This may suggest the conditions to choose in order to have strict inequality between the global and local Yamabe constant.

Finally, it is also important to remember that J. Viaclovsky has exhibited in [Via10] an example of an orbifold with one isolated singularity whose Yamabe constant coincides with the local one, and for which a Yamabe metric does not exist. It is then possible that other examples of non-existence occur, and it would be interesting to study (and classify?) them.

Outline of the thesis

Chapter 1 is devoted to geometric and analytic definitions concerning stratified spaces, to the examples, and to the presentation of various known results about Sobolev spaces and solutions to the Schrödinger equations in the singular setting. It is mainly based on [Klo09], [ALMP12], [ACM14] and [ACM15]. We also prove a regularity result for the gradient of a solution to the Schrödinger equation which is based on the spectral geometry of the links, and which represents a useful technical tool for many of the proofs in what follows.

In Chapter 2 we collect and prove the new results about admissible stratified spaces: in particular, one finds there the proofs of Theorems A, B and Corollary D.

Chapter 3 briefly presents the Yamabe problem on compact smooth manifolds and the results obtained in [ACM14] in the setting of stratified spaces. This part does not contain any new result, but it is important to motivate and understand what comes next.

Finally, the last Chapter is devoted to build up the connections between the results proven for admissible stratified spaces and the initial question of computing the local Yamabe constant. We prove there Corollary E and Theorem F. A second part of the chapter presents Theorem G with its consequence H. We conclude by the approach *à la* Obata with the proof of Theorem J and the reformulation of the theorem of N. Grosse for almost homogeneous manifolds.

Chapter 1

Stratified spaces

In this chapter we are going to present in details the setting of our results: we first give the definition of stratification and smooth stratification of a topological space, then we recall the main geometric properties and analytical tools that we will need in the following. We refer to [ALMP12], [ACM14] and [ACM15] for most of the definitions and the proofs.

1.1 Geometric Aspects of Stratified Spaces

The notion of stratification was originally introduced in topology by H. Whitney, with the idea of partitioning a topological space in simpler elements glued together in an appropriate way. If these elements are manifolds, we have roughly speaking a smooth stratification, and in particular we can consider Riemannian metrics on each element of it. For the sake of completeness, we briefly recall the definition of stratification from a topological point of view, by following the survey [Klo09] of B. Kloeckner. Next we describe the notion of smoothly stratified space as presented in [ALMP12] and [ACM14], together with some examples. We also give the definition of an admissible metric and a description of the local geometry with respect to this metric which follows [ACM15].

1.1.1 Definitions

The most general definition of stratification reads as follows: given a metrizable and separable topological space X, a stratification of X is a partition $\mathcal{S} = \{X^j\}_{j=0\dots N}$ whose elements X^j, called strata, are locally closed, and which satisfy for all $j_1, j_2 \in [0, N]$:

$$X^{j_1} \cap \bar{X}^{j_2} \neq \emptyset \quad \text{if and only if } X^{j_1} \subset \bar{X}^{j_2}.$$

We refer to the couple $(X, \mathcal{S})$ as a stratified space.
It is also possible to consider a decomposition of the form:

$$X = X_N \supset X_{N-1} \supset \dots \supset X_0 \tag{1.1}$$

where X_j, for $j = 0, \ldots N$, is closed. By defining $X^j = X_j \setminus X_{j-1}$ we obtain a partition as above with $\bar{X}^j = X_j$. Given a partition $\mathcal{S}$, we can always get a decomposition of the form (1.1). In order to explain how, we define the notion of topological depth:

Definition 1. Let $(X, \mathcal{S})$ be a stratified space. The topological depth of a stratum $X^j \in \mathcal{S}$ is the largest integer q such that there exists a chain $X^{j_0}, \ldots X^{j_q}$ such that $X^{j_q} = X^j$ and $X^{j_i} \subset \bar{X}^{j_{i-1}}$ for $i = 1, \ldots q$. We denote $q = \mathrm{depth}_t(X^j)$. The topological depth of X is the maximum among the topological depths of the strata:

$$\mathrm{depth}_t(X) = \max_{j=1,\ldots N} \mathrm{depth}_t(X^j).$$

Note that we are following the same convention for the inclusions $X^{j_i} \subset \bar{X}^{j_{i-1}}$ as in [ALMP12]. Given a stratified space $(X, \mathcal{S})$ of topological depth equal to $q \in \mathbb{N}$, we can define for any positive integer k less than or equal to q:

$$X_{q-k} = \bigcup_{\mathrm{depth}_t(X^j) \geq k} \bar{X}^j.$$

and obtain a decomposition of the form (1.1): $X_q \supset X_{q-1} \supset \ldots \supset X_0$ for X_j closed.

We are interested in compact *smoothly stratified spaces*: this means that the strata must be smooth manifolds of varying dimensions, and moreover that we require a more specific condition on how they are related. We will ask that for each stratum there exists a tubular neighbourhood with a precise structure, which is locally cone-like. We specify here that with a truncated cone $C(Z)$ over a topological space we mean the product $[0, 1] \times Z$ with the equivalence relation $(0, z_1) \sim (0, z_2)$ for all $z_1, z_2 \in Z$: this means that we identify all the points of $Z \times \{0\}$ to a unique point, the vertex of the cone.

Definition 2 (Smoothly Stratified Spaces)**.** Let X be a compact stratified space with stratification

$$\mathcal{S} = \{X^j\}_{j=1\ldots n}$$

We say that X is a smoothly stratified space if the following are satisfied:

1. Each stratum X^j is a smooth manifold of dimension j, possibly disconnected, X^{n-1} is empty and X^n is open and dense in X;

2. For each $j = 1 \ldots n$ and each connected component of X^j there exists a tubular neighbourhood $\mathcal{U}_j$ of X^j, a retraction π_j, a radial function ρ_j

$$\pi_j : \mathcal{U}_j \to X^j, \quad \rho_j : \mathcal{U}_j \to [0, 1].$$

   and a smoothly stratified space Z_j, such that π_j is a cone bundle, whose fibre in each point is a truncated cone over Z_j. The stratified space Z_j is called the link of the stratum X^j.

We also observe that the level set $\partial\mathcal{U}_j = \rho_j^{-1}(1)$ is the total space of a bundle $\pi_j : \partial\mathcal{U}_j \to X^j$ with fibre Z_j. We will not distinguish between this latter and the cone bundle of the definition.

By our definition, a smoothly stratified space contains an open dense subset which is a smooth manifold. We will refer to this dense subset, which is the stratum of maximal dimension X^n, as the regular set of X, and we will denote it Ω or X^{reg}. The dimension of a smoothly stratified space is then defined as the one of the regular set. Observe that as a consequence, for any $j = 1, \dots, n$ the dimension of the link Z_j must be equal to $d_j = n - j - 1$. With the singular set of X we mean $X \setminus \Omega$, and we denote it by Σ or X^{sing}.

Note that for a smoothly stratified space, we can easily obtain a decomposition of the form (1.1) without using the notion of topological depth but only the dimensions of the strata. In fact, if $(X, \mathcal{S})$ is a smoothly stratified space of dimension n we can define for any integer k less or equal than $(n-2)$

$$X_k = \bigcup_{\dim(X^j)\leq k} X^j, \quad X_n = X.$$

and we will have:

$$X_n \supset X_{n-2} \supset \dots \supset X_0. \tag{1.2}$$

The regular set of X will be then $X \setminus X_{n-2}$, and the singular set X_{n-2}. This is the notation that is used in [ACM15] and [Mon14]. Vice-versa, if we start with a decomposition of X based on the dimensions as in (1.2), we can get back to the Definition 2 by defining each stratum as $X^j = X_j \setminus X_{j-1}$.

We are going to describe more in detail the local structure of a stratified space X. By definition, for any $j = 1, \dots, n$ we have a cone bundle $\pi_j : \mathcal{U}_j \to X^j$. Thus for any point $x \in X^j$ there exist a trivializing neighbourhood $B^j(x)$ and a local continuous trivialization φ_x such that

$$\varphi_x : B^j(x) \times C(Z_j) \to \mathcal{W}_x = \pi_j^{-1}(B^j(x)). \tag{1.3}$$

satisfies $\pi_j \circ \varphi_x = \tilde{\pi}_1$, where $\tilde{\pi}_1$ is the projection on the first factor of the product $B^j(x) \times C(Z_j)$. For simplicity we can identify the neighbourhood $B^j(x)$ with an open ball in $\mathbb{R}^j$, by using local coordinates in X^j, so that we can summarize the preceding as follows: for each point $x \in X^j$ there exist a neighbourhood $\mathcal{W}_x$, a positive radius δ_x and an homeomorphism

$$\varphi_x : \mathbb{B}(\delta_x) \times C_{\delta_x}(Z_j) \to \mathcal{W}_x \tag{1.4}$$

where $\mathbb{B}(\delta_x)$ is a ball in $\mathbb{R}^j$ centred in 0 of radius δ_x, and $C_{\delta_x}(Z_j)$ is the truncated cone over Z_j of size δ_x; moreover φ_x restricts to a diffeomorphism on the regular sets, which are respectively $\mathcal{W}_x \cap \Omega$ and

$$\mathbb{B}(\delta_x) \times C_{\delta_x}(Z_j^{\mathrm{reg}}) \setminus \mathbb{B}(\delta_x) \times \{0\}.$$

This homeomorphism will be considered in the following as an identification between $\mathcal{W}_x$ and the product $\mathbb{B}^j(\delta_x) \times C_{\delta_x}(Z_j)$.

Let us introduce an iterative definition of depth which is more suitable to smoothly stratified spaces and follows the one given in Section 2 of [ACM14]. We underline that this notion of depth is *not* equivalent to the one given in Definition 2: this is the reason why we refer to the previous as *topological* depth instead of only depth, as it was done previously in the literature about stratified spaces.

Definition 3. Let X be a compact smooth manifold: then we define its depth to be equal to zero. Assume that for any positive integer k less or equal than $q > 0$ we have defined a smoothly stratified space of depth k, and consider Z a smoothly stratified space of depth equal to q. If X is a smoothly stratified space with one stratum whose link is Z, then we define its depth to be equal to :

$$\operatorname{depth}(X) = q + 1$$

More generally, if X has strata X^j, $j = 1, \dots n$, with links Z_j, we have:

$$\mathrm{d}(X) = 1 + \max_{\substack{j=1,\dots n \\ j \neq n-1}} \{\operatorname{depth}(Z_j)\}.$$

Note that we will often refer to the depth of a stratum as the depth of its link.

Roughly speaking, the depth tells us how much the local structure in cones is complicated, or how many steps we have to do to get to a smooth link. The previous definition can be be reformulated in the following way: the idea consists in decomposing a stratified space X into two subsets, one containing all the strata of maximal depth, and the other one *not* containing any of the strata of maximal depth.

Definition 4. Let $d \geq 0$ be an integer. We define the class $\mathcal{I}_d$ of smoothly stratified spaces of depth d iteratively:

1. An element of $\mathcal{I}_0$ is a compact smooth manifold;
2. A smoothly stratified space belongs to $\mathcal{I}_d$ if it can be decomposed in the union $X' \cup X''$, where X' is an element of $\mathcal{I}_{d-1}$ with a codimension one boundary along the intersection $X' \cap X''$. As for X'', each of its connected components is the total space of a bundle over a compact base, with fibre $C(Z)$ for some $Z \in \mathcal{I}_{d-1}$.

Before giving some examples, we briefly explain what we mean for a two-dimensional cone of angle α. In order to construct a cone we can imagine to have a circular sector of angle α, smaller that 2π, and identify the two radii that delimit it: we will simply have a cone of angle α, which is also a cone over a circle $\mathbb{S}^1$ of radius $a = \alpha/2\pi$ smaller than 1. If α is equal to 2π, we have clearly nothing to do. We also admit cones of angle α greater than 2π: in order to imagine the form that this cone takes, we can think of a plane, cut along a half-line, and we add a circular sector of angle β such that $\alpha = 2\pi + \beta$. Then we identify along the cut and get a cone of angle bigger that 2π, which corresponds to a cone over a circle $\mathbb{S}^1$ of radius greater than one. In both cases we will denote a two dimensional cone of angle $\alpha > 0$ as $C(\mathbb{R}/\alpha\mathbb{Z})$. Note that the cone of angle α has positive curvature in the sense of Alexandrov when α is less than 2π, and negative curvature in the sense of Alexandrov otherwise: we refer to the complete book [BBI01], and in particular to the example 4.1.4.

1.1.2 Examples

We first give three examples that belongs to the class of the manifolds with *simple edges*: these are nothing but stratified spaces whose links are compact smooth manifolds. In particular they have depth equal to one.

Conical singularities: The simplest example of compact smoothly stratified space is a manifold X of dimension n with isolated conical singularities $p_0, \dots p_k$. In this case we have only two strata: $X^n = X \setminus \{p_0, \dots p_k\}$ and $X^0 = \{p_0, \dots p_k\}$. The link of each conical point in X^0 is a compact manifold of dimension $(n-1)$. Surfaces with conical singularities arise very easily as quotients: for example, if we consider the sphere $\mathbb{S}^2$ and the rotation ξ of angle π around an axis, the quotient of $\mathbb{S}^2$ by the group generated by ξ is an American football, with two conical singularities at the South and North pole.

We give here an example in which conical singularities of angle bigger than 2π appear very easily.

A surface of genus two with conical singularities: Consider five (empty) cubes glued together in order to form a Greek cross and identify the upper face with the one on the right, and the lower face with the one on the left. We obtain a surface S of genus equal to two with a conical singularity at each of the eight vertices of the interior cube. The cone angle measures $5\pi/2$: in fact, the circular sector around an interior vertex comes from the three faces of the cross that contains it, two of them contributing for an angle of $\pi/2$ and the third one giving an angle of $3\pi/2$.

Pinched curve: We can have conical singularities along a curve instead of isolated ones. Consider a curve in a compact smooth manifold M^n of dimension $n = 3$ and a tubular neighbourhood of the curve. We can pinch this tubular neighbourhood at each point along the curve and create a singularity: a section of the tubular neighbourhood, that is a disk, is replaced by a cone over a circle. Therefore, each point of the curve has a neighbourhood which is the product of an interval $(-\varepsilon, \varepsilon)$, for $\varepsilon > 0$, with a cone $C(\mathbb{R}/\alpha\mathbb{Z})$ of angle $\alpha > 0$. Again we will have two strata, the pinched curve, and the regular set.

Manifolds with simple edges: We can imagine to apply a construction similar to the previous one for arbitrary dimension n, for a manifold M^n with boundary ∂M. We assume that the boundary is the total space for a bundle π with base B and fibre Z, for Z and B compact smooth manifolds: we can then obtain a bundle $\tilde{\pi}$ with total space $[0,1] \times \partial M$, same base and fibre $[0,1] \times Z$. We replace this latter with a cone bundle by gluing the fibres at $\{0\} \times Z$: this means that we identify via the equivalence relation $(0,x) \sim (0,y)$ if and only if $\tilde{\pi}(0,x) = \tilde{\pi}(0,y)$. Therefore we obtain a cone bundle with fibre $C(Z)$. In this case we refer to the singular space that we have constructed as a manifold with simple edges, that is a stratified space of depth equal to one, whose links are compact smooth manifolds.

Our definition includes spaces that are more singular than manifolds with simple edges, since we allow the link to be a stratified space as well. Note also that the previous examples have all depth equal to one. We describe four examples of depth equal to two that help to visualize a more singular situation.

Intersecting pinched curves: Consider a smooth manifold M^3 and two curves γ_1 and γ_2 intersecting transversally in one point p_0. If we repeat the construction above along the two curves, the point p_0 will give us a stratum of depth equal to two. In fact, a neighbourhood of p_0 will be a cone over a sphere $\mathbb{S}^2$ with four conical singularities, coming from the four points in which the neighbourhood intersects the curves γ_1 and γ_2. Therefore, we will have the regular set, the stratum of depth one given by the two curves without the point of intersection p_0, and the stratum of depth two consisting in p_0.

The double cube: Consider a solid cube C_1 and a copy C_2 of it: if we identify each face of C_1 with one face of C_2, we obtain a stratified space of depth equal to two. Each identified edge gives us a stratum of depth one along which the conical singularities have cone angle equal to π: this means that a neighbourhood of a point belonging only to one edge is the product of an interval with a cone over a circle of radius $1/2$. The eight identified vertices are a stratum of depth two, and a neighbourhood of one vertex is a cone over a sphere $\mathbb{S}^2$ having three conical singularities of angle π, coming from the three points in which a neighbourhood of a vertex intersects the edges.

The double cross: Consider five solid cubes glued together in order to form a Greek cross and denote this solid by G_1. Take a copy G_2 of it and as for the double cube identify each face of G_1 with the copied face in G_2. We have again a stratified space of depth equal to two: the identified edges give a stratum of depth one and the vertices a stratum of depth two. There is a difference between the interior vertices and edges of the cross, that is the eight vertices and four edges of the central cube, and the exterior ones. These latter behave like in the case of the double cube: along the exterior edges we have conical singularities of cone angle π, and each exterior vertex is the vertex of a cone over a sphere with three conical singularities of angle π. As for the interior edges, at each point we have to imagine that we glue together two circular sectors of angle $3\pi/2$: this means that at each point the cone angle is 3π. Therefore, a neighbourhood of an interior vertex is a cone over a sphere with three singularities, one of cone angle 3π and two of cone angle π.

Orbifolds: An n-orbifold with isolated singularities is a locally compact Hausdorff space M with singularities $\Sigma = \{(p_1, \Gamma_1), \dots, (p_s, \Gamma_s)\}$ such that $M \setminus \Sigma$ is a smooth manifold of dimension n, Γ_j is a finite subgroup of $O(n)$ acting freely on $\mathbb{R}^n \setminus \{0\}$ and there exist an open neighbourhood U_j and a homeomorphism $\varphi_j : U_j \to \mathbb{B}^n(0)/\Gamma_j$, for any $j = 1, \dots s$. This last quotient is homeomorphic to a cone over $\mathbb{S}^{n-1}/\Gamma_j$, so that we can see an orbifold as a stratified space with singular set Σ and of depth equal to two.

We can consider more in general orbifolds whose singularities are not isolated: there exists an atlas $(U_i, \tilde{U}_i, \varphi_i, \Gamma_i)$, where $\tilde{U}_i$ is an open set $\mathbb{R}^n$ on which the finite subgroup Γ_i acts, not necessarily freely, and φ_i is a homeomorphism between $\tilde{U}_i/\Gamma_i$ and U_i. See [Joy00], Section 6.5, for a more detailed definition.

Consider for simplicity an action of a finite group G in $O(n)$ on $\mathbb{R}^n$, and the quotient $\mathbb{R}^n/G$. By following the discussion in Section 9.1 of [Joy00], we know that a singular point vG of $\mathbb{R}^n/G$ is such that v is fixed by a subgroup Γ of G. Moreover, a neighbourhood of vG has the form of a product:

$$V^k \times W^{n-k}/\Gamma.$$

where V^k is the k-dimensional vector space of $\mathbb{R}^n$ which is fixed by Γ, and W^{n-k} is its orthogonal complement. This is clearly homeomorphic to $\mathbb{R}^k \times C(\mathbb{S}^{n-k-1}/\Gamma)$, and thus the quotient $\mathbb{R}^n/G$ has the structure of a stratified space. This discussion clearly holds for an n-orbifold, which is as a consequence a stratified space of depth at most equal to two.

1.1.3 Iterated edge metrics

The goal of this section is to define an admissible metric on a smoothly stratified space, with the basic idea that it must be a smooth Riemannian metric on the regular set behaving in the appropriate way near the singular strata: this means that an admissible metric must agree with the local structure given by the cone bundles. We refer to Section 3 of [ALMP12], where the existence of an admissible metric on a stratified space is proven, and to Section 2.1 of [ACM14].

We are going to give an iterative definition based on the depth of the space. If the depth is equal to zero, X is a compact smooth manifold and an admissible metric is clearly a Riemannian metric.

Let q be an integer and assume that we have defined an admissible metric on all smoothly stratified spaces of depth $q-1$. Consider a smoothly stratified space X of depth $d(X) = q$: then by definition of depth we can decompose X in X' and X'' in such a way that X' belongs to the class $\mathcal{I}_{q-1}$, and X'' contains the stratum X^j of maximal depth q. Then each connected component of X'' admits a neighbourhood $\mathcal{U}_j$ which is the total space of a cone bundle with fibre $C(Z_j)$, and Z_j is the link of depth $(q-1)$. By the iteration argument, an admissible metric exists on X' and on Z_j: it remains to describe the behaviour of the metric on each $\mathcal{U}_j$.

Consider $\partial\mathcal{U}_j = \rho_j^{-1}(1)$ and k a symmetric 2-tensor on $\partial\mathcal{U}_j$ which restricts to an admissible metric on each fibre of the bundle $\pi_j : \partial\mathcal{U}_j \to X^j$ and vanishes on a j-dimensional subspace. Let h be a smooth Riemannian metric on X^j. Then we define the *model metric* on $\mathcal{U}_j$ as:

$$g_0 = \pi_j^* h + d\rho_j^2 + \rho_j^2 k. \tag{1.5}$$

This means in particular that the metric induced by g_0 on each fibre of the cone bundle $\pi_j : \mathcal{U}_j \to X^j$ is an exact cone metric. An *admissible metric* g is then a metric defined by the inductive hypothesis away from $\mathcal{U}_j$ and which is a perturbation of the model metric on $\mathcal{U}_j$. This means that there exist $\gamma > 0$ and a positive constant C such that:

$$|g - g_0|_{g_0} \leq C\rho_j^\gamma \quad \text{on } \mathcal{U}_j.$$

Furthermore we require that $(g - g_0)$ vanishes at $\rho_j = 0$. To be more explicit, let X be a smoothly stratified space of depth q with strata X^j, $j = 1, \dots n$ and links Z_j, each of depth strictly inferior to q. For any $j = 1, \dots n$, let k_j be an admissible metric on Z_j and h_j a Riemannian metric on X^j. Then an admissible metric g on X is a smooth Riemannian metric on the regular set X^{reg} which near each stratum takes the form:

$$\pi_j^* h_j + d\rho_j^2 + \rho_j^2 k_j + E.$$

where E is a perturbation decaying as ρ_j^γ, $\gamma > 0$, as specified above.

We refer to g as an *admissible metric* or more often as an *iterated edge metric* on X. From now on we will denote by (X, g) a smoothly stratified space X endowed with iterated edge metric g, and we will refer to it simply as a stratified space.

Remark 1.1. Let $x \in X^j$ and consider local coordinates y on X^j and z on the link Z_j such that at x we have $\rho_j = 0$, $y = 0$ and $z = z_0$. Then the model metric at x is the product metric between the Euclidean metric on $\mathbb{R}^j$ and the exact cone metric $d\rho_j^2 + \rho_j^2 k_j$ on $C(Z_j)$.

Remark 1.2. The model metric g_0 is often referred as a *rigid* iterated edge metric in the literature: see for example [ALMP12], or Section 1 of [BV14]. In Section 3 of [ALMP12] it is also shown that any two admissible metrics on a stratified space are quasi-isometric.

A covering of X adapted to the metric

We can give a finite covering of (X, g) with open sets $\mathcal{W}_\alpha$ in such a way that on each $\mathcal{W}_\alpha$ the metric g is "not far" from the model metric.

More precisely, we know that for each point $x \in X^j$ we have a neighbourhood $\mathcal{W}_x$, and a radius $\delta_x > 0$ such that $\mathcal{W}_x$ is homeomorphic to $\mathbb{B}^j(\delta_x) \times C_{\delta_x}(Z_j)$. Since X is compact, we can choose a finite number of such open neighbourhoods $\mathcal{W}_\alpha$ which cover X. The covering is such that for each α there exist $\delta_\alpha > 0$, an open set $\mathcal{U}_\alpha$ in $\mathbb{R}^{d_\alpha}$, a compact stratified space Z_α of dimension $(n - d_\alpha - 1)$ and a homeomorphism φ_α between $\mathcal{W}_\alpha$ and $\mathcal{U}_\alpha \times C_{\delta_\alpha}(Z_\alpha)$. Moreover, we can construct the covering in such a way that there exists $\eta > 0$ such that for any radius $r < \eta$ and any point $x \in X$ the ball centred at x of radius r is included in at least one of the open sets $\mathcal{W}_\alpha$.

Each Z_α is endowed with a family of iterated edge metrics $\{k_\alpha(y), y \in \mathcal{U}_\alpha\}$, and $\mathcal{U}_\alpha$ carries a smooth Riemannian metric h_α. Then there exist positive constants Λ, γ such that for each α the metric g satisfies on $\mathcal{W}_\alpha$:

$$|\varphi^* g - (h_\alpha + dr^2 + r^2 k_\alpha)| \leq \Lambda r^\gamma.$$

where r is the radial function on the cone $C_{\delta_\alpha}(Z_\alpha)$.

Remark 1.3. It is possible to consider metrics of lower regularity: for example we can ask that h_α, and thus g on the regular set, is only γ-Hölder continuous. Most of the proofs we are going to present hold with this assumption as well, but for simplicity we only consider smooth metrics.

1.1.4 Tangent Cones

A useful tool to study stratified spaces is given by tangent cones, which generalize the notion of the tangent space. In fact, given a stratified space (X, g) with regular set Ω and singular set Σ, we cannot define the tangent space at any singular point x belonging to Σ. We consider instead the following definition:

Definition 5. Let X be a stratified space of dimension n, endowed with iterated edge metric g, and consider $x \in X$. The tangent cone at x is the unique Gromov-Hausdorff limit of the pointed metric spaces $(X, \varepsilon^{-2}g, x)$ as ε tends to zero.

If x belongs to the regular set Ω, then the tangent cone coincides with the usual tangent space and it is isometric to the Euclidean space $\mathbb{R}^n$.

If $x \in X^j$ for $j \neq n$, then the tangent cone is the cone $(C(S_x), dr^2 + r^2 h_x)$, where S_x is the $(j-1)$-fold spherical suspension of the link Z_j. More precisely, if $\mathbb{S}^{j-1}$ is the canonical sphere of dimension $(j-1)$ and $g_{\mathbb{S}^{j-1}}$ its canonical metric, we consider the product

$$\left(0, \frac{\pi}{2}\right) \times \mathbb{S}^{j-1} \times Z_j$$

endowed with the metric:

$$h_x = d\theta^2 + \sin^2\theta g_{\mathbb{S}^{j-1}} + \cos^2\theta k_j.$$

Then the $(j-1)$-fold spherical suspension S_x of Z is the completion of the previous product with respect to the metric h_x. We refer to S_x as the *tangent sphere* at x. Observe that if Z is not the round sphere, the tangent sphere S_x is a stratified space of dimension $(n-1)$ and depth equal to the one of Z plus one. In fact if we consider the metric completion of the product above, its singular set is constituted of the product:

$$\left(0, \frac{\pi}{2}\right) \times \mathbb{S}^{j-1} \times Z_x^{\text{sing}}.$$

which has the same depth as Z_x, plus the following two products:

$$\Sigma_0 = \{0\} \times \mathbb{S}^{j-1} \times Z_x, \text{ and } \Sigma_1 = \left\{\frac{\pi}{2}\right\} \times \mathbb{S}^{j-1} \times Z_x.$$

If we look at the metric near Σ_0, that is when θ is near to zero, we get that a point in Σ_0 has a neighbourhood of the form $\mathbb{B}^j \times Z_x$ (since $C(\mathbb{S}^{j-1}) = \mathbb{B}^j$), and therefore Σ_0 is a stratum of the same depth as Z. When we consider the metric for θ near to $\pi/2$, we obtain that a neighbourhood of a point in Σ_1 has the form $\mathbb{S}^{j-1} \times C(Z_x)$. As a consequence Σ_1 has depth equal to the one of Z plus one.

Note that $C(S_x)$ is a stratified space of dimension n with depth equal to the one of S_x plus one: therefore the depth of the tangent cone at x is equal to $\text{depth}(Z_x)$ plus two.

Lemma 1.4. *Let (X, g) be a stratified space of dimension n and let $x \in X^j$, $j \neq n$. Then the tangent cone $C(S_x)$ endowed with the exact cone metric $dr^2 + r^2 h_x$ is isometric to the product $\mathbb{R}^j \times C(Z_j)$ with the model metric $g_0 = dy^2 + d\tau^2 + \tau^2 k_j$, where dy^2 is the Euclidean metric on $\mathbb{R}^j$.*

Proof. It suffices to rewrite the Euclidean metric dy^2 in polar coordinates:

$$dy^2 = d\rho^2 + \rho^2 g_{\mathbb{S}^{j-1}}.$$

Then with the change of coordinates $\tau = r\sin(\theta)$, $\rho = r\cos(\theta)$ in g_0 we get $g = dr^2 + r^2 h_x$ where h_x coincides with the metric on the tangent sphere. This change of variables gives us an isometry between the regular sets of the two stratified spaces, that are $C(S_x^{\mathrm{reg}}) \setminus \{0\}$ and $\mathbb{R}^j \times C(Z^{\mathrm{reg}}) \setminus \{0\}$. This can be extended to the singular set by taking the completion of each of the two with the corresponding metric. $\square$

1.1.5 Geodesic balls

We follow now Section 2.2 in [ACM15] in order to describe the behaviour of geodesic balls in a stratified space (X, g) of dimension n. What it is possible to show, is that for an appropriate small radius, a geodesic ball is always contained in a truncated cone $C_\tau(S)$ over a connected stratified space S of dimension $(n-1)$. Moreover, on this ball the metric g differs from the exact cone metric on $C(S)$ by a factor τ^γ with $\gamma > 0$. The precise result is the following:

Proposition 1.5. *Let (X, g) a stratified space of dimension n and depth d, with covering $\{\mathcal{W}_\alpha\}_\alpha$ as introduced in the previous section. Let x be in X and α such that $x \in \mathcal{W}_\alpha$. There are positive constants Λ, η, κ such that $\Lambda\eta < 1$ and for any $\delta < \eta$ the ball $B(x, \delta)$ centred at x of radius δ is contained in $\mathcal{W}_\alpha$. Moreover, there exists a sequence of numbers $\varepsilon_0 > \varepsilon_1 > \ldots > \varepsilon_d = 0$ such that if we set $\tau_j = \delta \prod_{i=0}^{j-1} \varepsilon_i$ and τ belongs to the interval $[\tau_j, \tau_{j-1})$ the ball $B(x, \tau)$ is contained in an open set $\Omega_{x,\alpha}$:*

$$B(x, \tau) \subset \Omega_{x,\alpha} \subset B(x, 2\kappa\tau).$$

and $\Omega_{x,\alpha}$ is homeomorphic to a cone $C_{\kappa\tau}(S_{x,\alpha})$ over a connected stratified space $S_{x,\alpha}$ of dimension $(n-1)$. If we denote by $h_{x,\alpha}$ an admissible metric on $S_{x,\alpha}$ and $\varphi_{x,\alpha}$ the homeomorphism between $\Omega_{x,\alpha}$ and the cone over $S_{x,\alpha}$, we have on $\Omega_{x,\alpha}$:

$$|g - \psi^*(dt^2 + t^2 h_{x,\alpha})| \leq \Lambda \left(\frac{\tau}{\varepsilon_0 \varepsilon_1 \ldots \varepsilon_{j-1}} \right)^\gamma.$$

We briefly sketch the iterative argument based on the depth that constitutes the proof, and refer to [ACM15] for the details.

Assume first that X has depth equal to one. Then the link of each singular stratum X^j of X is a compact smooth manifold Z_j: assume for simplicity that there is only one connected singular stratum. Otherwise, we can repeat the same argument for the other strata and for each connected component. Let $\{\mathcal{W}_\alpha\}_\alpha$ and $\eta > 0$ be a covering and a radius as described above. Consider a point x in X and $\tau < \eta$: the ball $B(x,\tau)$ can be included in an open set $\mathcal{W}_\alpha$ not intersecting X^j, and in this case it is a geodesic ball in a smooth manifold, the regular set Ω. Otherwise, if $B(x,\tau)$ is included in one $\mathcal{W}_{\alpha,j}$ intersecting X^j, then its behaviour depends on the distance ε of x to X^j: roughly speaking, it depends on whether the ball is far enough from the singular stratum.

If τ is smaller enough than ε, then there exists a positive constant κ such that $B(x,\tau)$ is homeomorphic to the product:

$$\mathbb{B}^j(\kappa\tau) \times (\varepsilon - \kappa\tau, \varepsilon + \kappa\tau) \times B^Z(z_0, \kappa\tau/\varepsilon). \tag{1.6}$$

where $\mathbb{B}^j(\kappa\tau)$ is an Euclidean ball in $\mathbb{R}^j$, I is an interval and $B^Z(z_0, \kappa\tau/\varepsilon)$ a ball in the link Z centred at $z_0 \in Z$. Moreover, the metric g on $B(x,\tau)$ differs from the product metric on (1.6) for a an error decaying as $(\tau/\rho)^\gamma$, for γ as defined in section 1.1.3:

$$|g - (h_{\alpha,j} + dr^2 + \varepsilon^2 k)| \leq \Lambda_1 \left(\frac{\tau}{\varepsilon}\right)^\gamma. \tag{1.7}$$

where Λ_1 is some positive constant.

If τ is larger than ε, then it is possible to show that there exists a positive constant κ such that $B(x,\tau)$ is included in a ball centred at a singular point $x_0 \in X^j$ of radius $\kappa\tau$. A ball centred at x_0 is homeomorphic to a cone $C(S_{x_0})$ over the tangent sphere at x_0, and moreover the metric g on the ball differs from the cone metric on $C(S_{x_0})$ for an error decaying as τ^γ:

$$|g - (dr^2 + r^2 h_{x_0})| \leq \Lambda_2 \tau^\gamma. \tag{1.8}$$

This latter inequality proves the statement of Proposition 1.5 in the case of depth equal to one.

Now consider a stratified space X of depth equal to two. Then we can decompose it into two subsets X' and X'' in such a way that X' is a stratified space of depth equal to one and X'' contains all the strata of depth equal to two. If a ball $B(x,\tau)$ is included in X' then the previous discussion holds. Assume that $B(x,\tau)$ is included in X'' and in particular in an open set of the covering $\mathcal{W}_\alpha$ intersecting a stratum X^j, whose link is a stratified space Z_j of depth one. Let ε be the distance between x and X_j. Therefore, as above, $B(x,\tau)$ is either included in a cone with vertex x_0 in X^j over the tangent sphere S_{x_0}, and we have the desired estimate on the metric, or $B(x,\tau)$ is included in a product of the form (1.6). In this last case, the ball $B^{Z_j}(z_0, \kappa\tau/\varepsilon)$ is a ball in a stratified space of depth equal to one, and we have the previous description: either it is a geodesic ball completely contained in the regular set of Z_j, or its behaviour depends on the distance ε_1 of z_0 to the singular set of Z_j and on whether the radius $\kappa\tau/\varepsilon$ is smaller or larger than ε_1. When the radius is larger than ε_1, by an argument similar to the one explained in the previous section, the product (1.6) is homeomorphic to the cone $C(\hat{S}_{z_1})$ over the j-spherical suspension of S_{z_1}, for a point z_1 in the singular set of Z_j with distance from z_0 equal to ε_1. Furthermore, thanks to the estimates (1.7) and (1.8), we get that the metric g on the ball $B(x,\tau)$ differs from the cone metric on $C(\hat{S}_{z_0})$ for an error of size $(\tau/\varepsilon)^\gamma$. This explains why in Proposition 1.5 the numbers $\varepsilon_0, \dots \varepsilon_j$, which depend on the depth, appear in the estimate for the metric.

1.2 Analytical Aspects of Stratified Spaces

This section is devoted to the definition of some analytical tools on stratified spaces, in particular Sobolev spaces and the Laplacian operator. We collect here part of the results contained in [ACM14] and [ACM15] which we will extensively use later. In a first part we review how some issues about analysis on compact manifolds, like Sobolev embeddings, naturally extend to stratified spaces. In a second part we recall some results about the Schrödinger equation on stratified spaces, since we will often encounter this kind of equation in the next chapters. We finally give an improvement of a proposition contained in [ACM15], under one additional assumption, which will reveal to be a useful tool in many of the proofs presented in this thesis.

1.2.1 Sobolev spaces and inequalities

Given a stratified space (X, g) of dimension n, we denote by dv_g the measure associated to the iterated edge metric g: such measure is Ahlfors n-regular, that is there exists a positive constant C such that for any $x \in X$ and $0 < r < \operatorname{diam}(X)/2$ the measure of the ball $B(x, r)$ of radius r centred at x is bounded between:

$$C^{-1} r^n < \mathrm{Vol}_g(B(p, r)) < C r^n.$$

Furthermore, as a consequence of the Ahlfors regularity, dv_g is a doubling measure: there exists a constant C_1 such that for any $x \in X$ and $0 < r < \operatorname{diam}(X)/2$ we have:

$$\mathrm{Vol}_g(B(x, 2r)) \leq C_1 \mathrm{Vol}_g(B(x, r)).$$

We are going to define classical analytic tools on (X, g): the space $L^p(X)$ for $p \in [1, +\infty)$ is obviously defined as the equivalence classes of the functions f whose power p has finite integral on X, with respect to the relation $f \backsim g$ if and only if $f = g$ almost everywhere. The space $L^\infty(X)$ is the space of bounded functions on X up to the same equivalence relation. The norms in the spaces $L^p(X)$ are the usual ones.

We are interested in Sobolev spaces, which are defined as follows:

Definition 6. Let (X, g) be a stratified space of dimension n, and $p \in (1, +\infty]$. The Sobolev space $W^{1,p}(X)$ is the completion of the space of Lipschitz functions $\mathrm{Lip}(X)$ with respect to the norm:

$$\|f\|_{1,p}^p = \|f\|_p^p + \|df\|_p^p.$$

For the moment we restrict our discussion to the case $p = 2$. The assumption that there are no strata of codimension one allows us to prove the following:

Lemma 1.6. *Let (X, g) be a smoothly stratified space and Ω the regular set of X. Then the space $C_0^1(\Omega)$ of continuously differentiable functions with compact support in Ω is dense in $W^{1,2}(X)$.*

The proof consists in choosing the appropriate Lipschitz cut-off functions with compact support in the regular set. We sketch this argument here for two reasons: it motivates the necessity of avoiding strata of codimension equal to one, and it allows us to introduce some cut-off functions that will be widely used later.

Let $\varphi \in \mathrm{Lip}(X)$. We aim to find a sequence of functions in $\mathrm{Lip}_0(\Omega)$ which converges in the norm of $W^{1,2}(X)$ to φ. This is enough to prove our statement, because for any function f in $\mathrm{Lip}_0(\Omega)$ we can find a sequence of smooth functions with compact support in Ω converging to f in $W^{1,2}(X)$. This is done by using a standard argument based on local charts, partitions of unity and convolution in $\mathbb{R}^n$. In fact, we can cover the compact support of f with a finite number of local charts (U_i, ψ_i) of Ω, and then associate a partition of unity χ_i to this covering, so that we can rewrite:

$$f_i = \sum_{i=1}^{k} \chi_i f.$$

By convolution with mollifiers in $\mathbb{R}^n$, we can find a sequence of smooth functions with compact support in $\mathbb{R}^n$ converging to $(\chi_i f) \circ \psi_i^{-1}$ for each $i = 1, \dots k$. Pulling back these sequences to Ω gives us the desired sequence of functions in $C_0^\infty(\Omega)$ converging to f in the norm of $W^{1,2}(X)$. We refer to Proposition 2.4 and Theorem 2.4 in [Heb99] for more precise details.

It remains to show that the Lipschitz functions with compact support in the regular set are dense in $\mathrm{Lip}(X)$. Recall that the singular set Σ of X consists of a finite number of connected components Σ^j, with possibly different dimensions: we denote by $m \geq 2$ the minimal codimension of the strata Σ^j, and by t the distance function from the singular set $t(x) = d_g(x, \Sigma)$. We are going to choose cut-off functions depending on whether $m > 2$ or $m = 2$.

Assume first that $m = 2$ and for a small $\varepsilon > 0$ consider the tubular neighbourhoods of size ε and ε^2: we denote them by Σ^ε and Σ^{ε^2}. We consider

$$f_\varepsilon(t) = \begin{cases} 1 & \text{in } X \setminus \Sigma^\varepsilon \\ 2 - \dfrac{\ln(t)}{\ln(\varepsilon)} & \text{in } \Sigma^\varepsilon \setminus \Sigma^{\varepsilon^2} \\ 0 & \text{in } \Sigma^{\varepsilon^2}. \end{cases} \qquad |df_\varepsilon| = \frac{1}{t|\ln(\varepsilon)|}.$$

Then f_ε is a Lipschitz function with compact support in Ω. Furthermore, we claim that there exists a positive constant A such that we have:

$$\left(\int_{\Sigma^\varepsilon \setminus \Sigma^{\varepsilon^2}} |df_\varepsilon|^2 dv_g \right) \leq \frac{A}{|\ln(\varepsilon)|}. \tag{1.9}$$

Let us assume that $-\ln(\varepsilon)$ is an integer number N. Then we can decompose $\Sigma^\varepsilon \setminus \Sigma^{\varepsilon^2}$ in the disjoint union:

$$\Sigma^\varepsilon \setminus \Sigma^{\varepsilon^2} = \bigcup_{j=N}^{2N-1} \Sigma^{e^{-j}} \setminus \Sigma^{e^{-(j+1)}}.$$

As a consequence the integral (1.9) can be written as the following sum:

$$\begin{aligned}\left(\int_{\Sigma^{\varepsilon}\setminus\Sigma^{\varepsilon^2}} |df_{\varepsilon}|^2 dv_g\right) &= \frac{1}{|\ln(\varepsilon)|^2}\left(\int_{\Sigma^{\varepsilon}\setminus\Sigma^{\varepsilon^2}} \frac{1}{t^2} dv_g\right)\\
&= \frac{1}{|\ln(\varepsilon)|^2}\sum_{j=N}^{2N-1}\int_{\Sigma^{e^{-j}}\setminus\Sigma^{e^{-(j+1)}}} \frac{1}{t^2} dv_g\\
&\leq \frac{1}{|\ln(\varepsilon)|^2}\sum_{j=N}^{2N-1}\int_{\Sigma^{e^{-j}}\setminus\Sigma^{e^{-(j+1)}}} e^{2(j+1)} dv_g\\
&\leq \frac{1}{|\ln(\varepsilon)|^2} A(N-1) \leq \frac{A}{|\ln(\varepsilon)|}.\end{aligned}$$

which is the estimate we wanted to prove. In the last line we used that the volume of a tubular neighbourhood of size ε of Σ is smaller than a constant times ε^m, and in this case $m = 2$.

If $m > 2$ consider the tubular neighbourhoods Σ^{ε} and $\Sigma^{2\varepsilon}$ and define the following function:

$$g_{\varepsilon}(t) = \begin{cases} 1 \text{ on } X \setminus \Sigma^{2\varepsilon} \\ \dfrac{t}{\varepsilon} - 1 \text{ on } \Sigma^{2\varepsilon} \setminus \Sigma^{\varepsilon} \\ 0 \text{ on } \Sigma^{\varepsilon}. \end{cases} \qquad |dg_{\varepsilon}| = \frac{1}{\varepsilon}.$$

This is again a Lipschitz function with compact support in Ω. A straightforward computation shows that there exists a constant $B > 0$ such that:

$$\int_{\Sigma^{2\varepsilon}\setminus\Sigma^{\varepsilon}} |dg_{\varepsilon}|^2 dv_g \leq \int_{\Sigma^{2\varepsilon}\setminus\Sigma^{\varepsilon}} \frac{1}{\varepsilon^2} dv_g \leq B\varepsilon^{m-2}.$$

In both cases we have found a cut-off function in $\mathrm{Lip}_0(\Omega)$ such that the norm in $L^2(X)$ of the gradient converges to zero as ε tends to zero. Now consider the functions $f_{\varepsilon}\varphi$ if $m = 2$ and $g_{\varepsilon}\varphi$ otherwise, which belong to $\mathrm{Lip}_0(\Omega)$. When ε tends to zero, $f_{\varepsilon}\varphi$ and $g_{\varepsilon}\varphi$ converge in the Sobolev space $W^{1,2}(X)$ to φ: this is an easy computation due to the fact that f_{ε} and g_{ε} converge to the constant function equal to one in $L^{\infty}(X)$, and to Hölder inequality applied to the terms φdf_{ε} and φdg_{ε} for what concerns the norm in $L^2(X)$ of the gradients.

Observe that if we consider the Sobolev space $W^{1,p}(X)$ for p greater than two, and if we want $C_0^1(\Omega)$ to be dense in it, we need to assume that the stratified space has only strata of codimension larger than or equal to p. We give another simple and useful lemma about the composition of a function u in $W^{1,2}(X)$ with a real-valued function.

Lemma 1.7. *Let $u \in W^{1,2}(X)$ and $f \in C^1(\mathbb{R})$. If the first derivative of f is bounded, then $f \circ u$ belongs to $W^{1,2}(X)$.*

The Laplacian operator on a stratified space is defined as follows, in terms of generator of a quadratic form:

Definition 7. Let (X, g) be a stratified space. Consider the semi-bounded quadratic form $\mathcal{E}$ defined on $C_0^\infty(\Omega)$ by:

$$\mathcal{E}(u) = \int_X |du|^2 dv_g.$$

Let Δ_g be the self-adjoint operator obtained as Friedrichs extension of $\mathcal{E}$. We refer to Δ_g as the Laplacian operator associated to g.

We have chosen the definition of Δ_g in such a way that it is a non-negative operator: this means that in the Euclidean space we would have taken as Laplacian operator the one with minus sign $-\sum_i \partial_i^2$.

We are now in position to state that the Sobolev inequality holds on a stratified space. This was proven in [ACM14], together with some of its consequences: the compactness of the Sobolev embeddings and the discreteness of the spectrum of the Laplacian. We recall these results and briefly sketch their proofs.

Proposition 1.8 (Sobolev inequality). *Let (X, g) be a smoothly stratified space of dimension n. Then there exists a positive constant C_s such that for any function u in the Sobolev space $W^{1,2}(X)$ we have:*

$$\left(\int_X |u|^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{n}} \leq C_s \int_X (|u|^2 + |du|^2) dv_g.$$

The proof is by iteration on the depth of the space (X, g). If the depth is equal to zero, then (X, g) is a smooth compact manifold, and the existence of a Sobolev inequality is a well-known result in this case (see for example [Heb99]). Assume that we have proven the proposition for all depths smaller than or equal to $(d-1)$, for an integer $d \geq 1$, and consider a stratified space X of depth d. Then X can be decomposed into the union of X' belonging to the class of stratified spaces of depth $(d-1)$ and X'' containing the stratum of maximal depth d. Assume for simplicity that X'' is connected (and repeat otherwise the following discussion for each connected component of X''): each point of X'' admits a neighbourhood of the form $\mathbb{B}^j \times C(Z)$ for an Euclidean ball $\mathbb{B}^j$ in $\mathbb{R}^j$ and a compact stratified space Z of depth $(d-1)$, endowed with an iterated edge metric k. Observe also that the Sobolev inequality is invariant under quasi-isometric changes of the metric, so that we can prove it by using the model metric g_0 introduced in Section 1.3. Then it suffices to prove the Sobolev inequality on $\mathbb{B}^j \times C(Z)$ endowed with the metric $dy^2 + dr^2 + r^2 k$.

The Sobolev inequality is in fact equivalent to a diagonal upper bound on the heat kernel P associated to the Laplacian operator:

$$P(t, x, x) \leq Ct^{-\frac{n}{2}} \quad \text{for all } t \in \mathbb{R}, x \in X.$$

This estimate is true in the Euclidean space $\mathbb{R}^j$ and in Z as well, since by the iterative assumption the Sobolev inequality holds on Z. Moreover, the heat kernel of a product

$X \times Y$ is the product of the heat kernels P_X and P_Y: is suffices then to show than a diagonal upper bound holds on the cone $C(Z)$ endowed with the metric $\bar{k} = dr^2 + r^2k$. The next step of the proof consists in decomposing $C(Z)$ into conic slices of the appropriate size, on which the heat kernel estimates holds, and in gluing them together by means of a partition of the unity. This leads to the following inequality:

$$\|u\|^2_{L^{\frac{2n}{n-2}}(C(Z))} \leq C \left(\|du\|^2_{L^2(C(Z))} + \int_{C(Z)} \frac{u^2}{r^2} dv_{\bar{k}} \right). \tag{1.10}$$

In order to handle the last term of this inequality, one can use separation of variables on the cone and the spectral decomposition of the Laplacian on $C(Z)$, since by the iteration hypothesis the Laplacian on Z has discrete spectrum. See Corollary 1.10 for an explanation of how the Sobolev inequality implies the discreteness of the spectrum of the Laplacian. This allows one to get a Hardy inequality on $C(Z)$ and to bound the last term in (1.10) by a constant times the norm of the gradient in $L^2(C(Z))$. Then the following holds on the cone:

$$\|u\|^2_{L^{\frac{2n}{n-2}}(C(Z))} \leq C\, \|du\|^2_{L^2(C(Z))} \leq C\, \|u\|^2_{W^{1,2}(C(Z))}.$$

This leads to the desired diagonal upper bound on the heat kernel, and therefore to the Sobolev inequality.

As a consequence of Proposition 1.8, it is possible to prove the compactness of the Sobolev embeddings:

Proposition 1.9. *Let (X, g) be a smoothly stratified space of dimension n. The inclusion $W^{1,2}(X) \hookrightarrow L^q(X)$ is compact for any $q \in \left[1, \frac{2n}{n-2}\right)$.*

Observe that in order to prove this statement, it suffices to show the compactness of the embedding with q equal to $\frac{2p}{p-2}$ for $p \in (n, \infty)$. In fact we have:

$$2 < q < \frac{2n}{n-2}, \quad L^q(X) \hookrightarrow L^2(X) \hookrightarrow L^1(X).$$

Then by composition of a continuous embedding with a compact one, we get the compactness for $q = 1, 2$ as well. The proof consists in approximating the embedding $\imath : W^{1,2}(X) \hookrightarrow L^q(X)$ by a sequence of compact operators converging to $\imath$ in the operator norm. Such sequence is given by the semi-group of operators associated to the quadratic form

$$Q(u, u) = \int_X (|du|^2 + |u|^2) dv_g.$$

For $t > 0$ we denote an element of this semi-group by $T_t = e^{-t(\Delta+1)}$. It is then necessary to get an estimate the norm of $(u - T_t u)$ in $L^q(X)$, holding for any u in $W^{1,2}(X)$, and to show that this norm tends to zero as t goes to zero. In order to prove this, it is possible to use estimates of the norm of T_t as an operator from $L^r(X)$ to $L^r(X)$ with the appropriate exponent r, and as an operator from $L^1(X)$ to $L^\infty(X)$. These estimates, together with the spectral theorem in the form of functional calculus, give the desired bound for the norm of $(u - T_t u)$ in $L^q(X)$: there exists a positive constant C such that

$$\|u - T_t u\|_q \leq C t^{\frac{1}{2} - \frac{n}{2p}}.$$

Letting t tend to 0 concludes the proof.

The compact Sobolev embeddings allow one to prove that the spectrum of the Laplacian operator is discrete.

Corollary 1.10. *The Laplacian Δ_g has discrete spectrum.*

In fact, if we denote by $\mathcal{D}(\Delta_g)$ the domain of the Laplacian and consider the operator:

$$(\Delta_g + 1)^{-1} : L^2(X) \to \mathcal{D}(\Delta_g),$$

this is a self-adjoint and bounded operator (its norm is smaller than 1). Moreover, $\mathcal{D}(\Delta_g)$ is equal to $W^{2,2}(X) \cap W_0^{1,2}(X) \subset W^{1,2}(X)$, which is compactly contained in $L^2(X)$, therefore $(\Delta_g + 1)^{-1}$ is also compact. Thanks to the spectral theorem for compact self-adjoint operators, $(\Delta_g + 1)^{-1}$ has discrete spectrum, and the same holds for $(\Delta_g + 1)$ and thus obviously for Δ_g.

In [ACM15] the authors also prove that a scale-invariant Poincaré inequality holds on a stratified space. This can furnish an alternative proof of the Sobolev inequality, since a scale-invariant Poincaré inequality on an almost smooth metric space with doubling measure implies the Sobolev inequality.

Proposition 1.11 (Poincaré Inequality)**.** *Let (X, g) be a stratified space of dimension n. Assume that for each $x \in X$ the tangent sphere S_x is connected. There exist constants $a > 1$ and $C, \rho_0 > 0$ such that if B is a ball of radius $\rho < \rho_0$ centred at x, and aB the ball of radius $a\rho$ centred at the same point, then for any function $u \in W^{1,2}(aB)$ we have a scale-invariant Poincaré inequality:*

$$\int_B |u - u_B|^2 dv_g \leq C\rho^2 \int_{aB} |du|^2 dv_g.$$

where $u_B = \mathrm{Vol}_g(B)^{-1} \int_B u dv_g$.

The proof can be done by iteration on the dimension of the space and a localization argument. Assume that the Poincaré inequality is proven for all stratified space of dimension smaller than or equal to $(n-1)$, and consider a stratified space X of dimension n. By the previous discussion about geodesic balls, we know that there exists ρ_0 such that for any x in X and a sufficiently small radius $\rho < \rho_0$, the ball $B(x, \rho)$ is contained in an open set which is homeomorphic to a cone $C(S)$ over connected stratified space S of dimension $(n - 1)$. Moreover, the metric g differs from the exact cone metric $g_0 = dr^2 + r^2 h$ on $C(S)$ for a factor ρ^γ, $\gamma > 0$. Therefore, it suffices to show that the Poincaré inequality holds on the cone $C(S)$, and this is done by using the iterative assumption, separation of variables and the spectral decomposition of the Laplacian associated to g_0.

1.2.2 Solutions to the Schrödinger equation

In the following, we often need to study solutions of a Schrödinger equation on a stratified space (X, g) of dimension n. Given V belonging to some $L^q(X)$ for $q \in [1, \infty]$, we consider the Schrödinger operator $\Delta_g - V$. A weak solution $u \in W^{1,2}(X)$ of the equation

$$\Delta_g u - Vu = 0.$$

is such that for any $\varphi \in W^{1,2}(X)$, with $\varphi \geq 0$, we have:

$$\int_X (d\varphi, du) dv_g = \int_X V\varphi u dv_g.$$

In order for this equality to make sense, we need that V belongs to $L^q(X)$ for $q \geq n/2$: in this case, thanks to the Sobolev embedding, we can apply the Hölder inequality to the right-hand side and get that it is finite.

The first result about weak solutions is that, if V belongs to the appropriate $L^q(X)$, then u is bounded: this is done by using Moser iteration technique, which we will apply several times in different contexts. We refer to Proposition 1.8 in [ACM14]:

Proposition 1.12 (Moser Iteration technique)**.** *Let (X, g) be a stratified space of dimension n and V in $L^q(X)$ for $q > \frac{n}{2}$. Assume that $u \in W^{1,2}(X)$ is non-negative and it satisfies the weak inequality:*

$$\Delta_g u - Vu \leq 0.$$

Then u is bounded on X and there exists a positive constant C such that:

$$\|u\|_\infty \leq C \|u\|_2 .$$

We recall the main steps of the proof. Since u belongs to $W^{1,2}(X)$, the Sobolev embeddings assure that u is in $L^q(X)$ for $q \in \left[1, \frac{2n}{n-2}\right]$. Moser iteration technique consists in using the Sobolev inequality and the weak equation (or inequality) satisfied by u with a good test function, in order to gain more regularity on u: this means to increase the exponent q such that u belongs to $L^q(X)$. The good test function appears to be a power of u, but we have to be careful: for $\alpha > 1$ the function u^α is not necessarily in $W^{1,2}(X)$, since $f(x) = x^\alpha$ is clearly $C^1(\mathbb{R})$, but its derivative is not bounded (see Lemma 1.7). As a consequence of this, it is necessary to approximate $f(x) = x^\alpha$ by a continuously differentiable function $f_{\alpha,L}$ which coincides with x^α on larger an larger intervals $[0, L]$, and which is a line of slope equal to one elsewhere. By combining the Sobolev inequality applied to $f_{\alpha,L} \circ u$, the weak inequality with a test function obtained from $f_{\alpha,L}$, and the Hölder inequality with exponent q, it is possible to get for some positive constant C_1:

$$\left(\int_X f_{\alpha,L}(u)^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{n}} \leq C_1 \alpha \left(\int_X f_{\alpha,L}(u)^{\frac{2q}{q-1}} dv_g\right)^{\frac{q-1}{q}}.$$

Therefore, if we denote:

$$r = \frac{2q}{q-1} < \frac{2n}{n-2} \quad \text{and} \quad \gamma = \frac{q-1}{q}\frac{n}{n-2} > 1.$$

and we let L tend to infinity, the previous inequality leads to:

$$\|u\|_{\alpha\gamma r} \leq (C_1\alpha)^{\frac{1}{2\alpha}} \|u\|_{\alpha r} \tag{1.11}$$

Choose α close enough to 1, in such a way that we have $\alpha r \leq \frac{2n}{n-2}$. The Sobolev space $W^{1,2}(X)$ is contained in $L^q(X)$ for any q between 1 and $\frac{2n}{n-2}$, then it is in particular contained in $L^{\alpha r}(X)$. Therefore with this choice of α the right-hand side of (1.11) is finite, and as a consequence the left-hand side is finite as well. Since $\alpha\gamma r$ is strictly greater than αr, the regularity of u has been increased. It is now possible to iterate the same argument leading to (1.11) for $\alpha_j = \gamma^j\alpha$ for an integer j.

As for the first step of the iteration, consider $\alpha_2 = \gamma^2\alpha$: then by repeating the previous argument with α_2 we get

$$\|u\|_{\gamma^2\alpha r} \leq (C_2\gamma\alpha)^{\frac{1}{2\gamma\alpha}} \|u\|_{\gamma\alpha r} \leq (C_2\gamma\alpha)^{\frac{1}{2\gamma\alpha}} (C_2\alpha)^{\frac{1}{2\alpha}} \|u\|_{\alpha r}.$$

By iterating the same argument N times, for $\alpha_N = \gamma^N\alpha$, one obtains:

$$\|u\|_{\alpha_N r} \leq \prod_{j=0}^{N-1} (C_2\gamma^j\alpha)^{\frac{1}{2\gamma^j\alpha}} \|u\|_{\alpha r}$$

When N tend to infinity, the left-hand side of the previous inequality converges to the norm of u in $L^\infty(X)$. The constant in the right-hand side converges to a finite number as well. Besides, for some positive C' we have that $\|u\|_{\alpha r} \leq C' \|u\|_{1,2}$. Therefore, there exists a positive constant C such that:

$$\|u\|_\infty \leq C \|u\|_{1,2}. \tag{1.12}$$

In order to replace the norm in $W^{1,2}(X)$ of u with its norm in $L^2(X)$ it suffices to use again the weak inequality with test function equal to u and interpolation between the norm in $L^2(X)$ and $L^\infty(X)$. In fact we have:

$$\|du\|_2^2 = \int_X u\Delta_g u dv_g \leq \int_X Vu^2 dv_g \leq \|V\|_{\frac{n}{2}} \|u\|^2_{\frac{2n}{n-2}} \leq \|V\|_{\frac{n}{2}} \|u\|_2^{2(1-\theta)} \|u\|_\infty^{2\theta}$$

where $\theta \in (0,1)$. Now by Young inequality we obtain for any $\varepsilon > 0$:

$$\|du\|_2^2 \leq \frac{1-\theta}{\varepsilon} \|V\|_{\frac{n}{2}}^{\frac{1}{1-\theta}} \|u\|_2^2 + \theta\varepsilon \|u\|_\infty^2.$$

Therefore if one chooses ε sufficiently small (say $\varepsilon < 1/C^2\theta$) and replace in (1.12), the conclusion of Proposition 1.12 easily follows.

Remark 1.13. The previous proposition applies in particular to the equation

$$\Delta_g u = u^{q-1}$$

for any q which is strictly smaller that the critical exponent in the Sobolev compact embeddings, that is $p = 2n/(n-2)$, and $u > 0$. This can be seen by showing that u to the power $(q-2)$ belongs to $L^{\alpha n/2}(X)$ for some α larger than one: the first step of the previous proof and the Hölder inequality with the appropriate exponents prove indeed that u is integrable to the power $2\alpha p$.

It is more difficult to get a better regularity on a solution to the Schrödinger equation, for example Hölder regularity: we recall here a result contained in [ACM15] that gives a strong relation between the Hölder exponent and the spectral geometry of the links. More precisely, if we denote by $\lambda_1(Z)$ the first non zero eigenvalue of the Laplacian on the link Z, the exponent ν such that a solution u to the Schrödinger equation belongs to $C^{0,\nu}(X)$ is determined by the infimum of $\lambda_1(Z)$ over the links.

We have recalled in the previous section that on a stratified space (X, g) of dimension n, the Laplacian Δ_g associated to the iterated edge metric g has discrete spectrum: we denote by $\lambda_1(X)$ its first non zero eigenvalue. We can alternatively write for any λ_j eigenvalue of Δ_g:

$$\lambda_j = \nu_j(n-1+\nu_j).$$

for a unique value ν_j in the interval $(0,1)$. Then we define:

$$\nu_1(X) = \begin{cases} 1 & \text{if } \lambda_1(X) \geq n, \\ \nu_1 & \text{if } \lambda_1 < n. \end{cases}$$

The case in which we will be the most interested is when $\lambda_1(X)$ is greater than or equal to the dimension of the space, but for the sake of completeness we state the general Hölder regularity result: we refer to Theorem A in [ACM15].

Theorem 1.14 (Hölder regularity). *Let (X, g) be a compact stratified space of dimension n. For each $x \in X$, denote by Z_x the link of the cone bundle over the stratum containing x, and define:*

$$\nu = \inf_{x \in M} \nu_1(Z_x).$$

Let u in $W^{1,2}(X)$ be a solution to the Schrödinger equation $\Delta_g u - Vu = 0$ for V belonging to $L^p(X)$, $p > n/2$. Then we have:

1. *If $p = \infty$ and $\nu = 1$, then there is a positive constant C such that for all $x, y \in X$ with $d_g(x, y) \leq 1/2$:*

$$|u(x) - u(y)| \leq C\sqrt{|\log(d_g(x,y))|}d_g(x,y).$$

2. *If $p = \infty$ and $\nu \in (0,1)$, then u belongs to $C^{0,\nu}(X)$.*

3. *If $p \in (n/2, \infty)$ and $\nu \in (0,1]$, then u belongs to $C^{0,\mu}(X)$, where*

$$\mu = \min\left\{\nu, 1 - \frac{n}{2p}\right\}.$$

For any $x \in X$, the first non zero eigenvalue $\lambda_1(Z_x)$ of the Laplacian on the link determines the first non zero eigenvalue of the Laplacian on the tangent sphere S_x. In particular we have for any x in X:

$$\nu_1(Z_x) = \nu_1(S_x) \quad \text{for any } x \in X. \tag{1.13}$$

Therefore, the hypothesis on the link can be replaced by an hypothesis on the tangent sphere. This can be easily proven by using separation of variables and the spectral decomposition of the Laplacian operator on the $(j-1)$-fold spherical suspension S_x of Z_x: we refer to Section 3.6 in [ACM15] for the details.

The proof of Theorem 1.14 relies on two results: the first one establishes the relation between a Morrey inequality and the Hölder regularity of a function. The second one (Proposition 4.1 in [ACM15]) allows one to deduce, by a local argument, the appropriate Morrey inequality that a solution u to the Schrödinger equation satisfies: this is proven through an interesting study of Dirichlet to Neumann operators.

We state here a consequence of 1.14, contained in [Mon14] that gives us a useful technical tool for many of the proofs presented in this thesis. In the case $\nu = 1$ and $p = +\infty$, let u be a solution to the Schrödinger equation $\Delta_g u = Vu$: under a further assumption on the gradient du, we can give an estimate for the norm of du in $L^\infty(X)$ away from the singular set Σ and depending from the distance to Σ. For $\varepsilon > 0$, let us denote by Σ^ε the tubular neighbourhood of size ε of Σ.

Proposition 1.15. *Let (X^n, g) be a stratified space and assume that for any x we have*

$$\lambda_1(Z_x) \geq \dim(Z_x).$$

or, equivalently, $\nu_1(S_x) = 1$. Let V be in $L^\infty(X)$ and u be a solution to the equation $\Delta_g u - Vu = 0$. Assume that there exists a positive constant c such that

$$\Delta_g |du| \leq c|du| \text{ on } \Omega. \tag{1.14}$$

Then there exists a positive constant C such that for any $\varepsilon > 0$ we have away from Σ^ε:

$$\|du\|_{L^\infty(X \setminus \Sigma^\varepsilon)} \leq C\sqrt{|\ln(\varepsilon)|}. \tag{1.15}$$

The proof of this result consists in combining Theorem 1.14 and the Moser iteration technique, which allows us to estimate the gradient on a ball centred at a regular point with the mean of its norm in $L^2(X)$ over a bigger ball.

Lemma 1.16. *Let (X, g) be a stratified space of dimension n and $f \in L^2(X)$ such that for some positive constant c the strong inequality*

$$\Delta_g f \leq cf. \tag{1.16}$$

holds on Ω. Then there exists a positive constant c_1, depending on c, on the dimension n and on the Sobolev constant C_s, such that for any regular point $x \in \Omega$ and radius $0 < r < d_g(x, \Sigma)/2$ we have:

$$\|f\|_{L^\infty(B(x,r/2))} \leq c_1 \left(\frac{1}{r^n} \int_{B(x,3r/4)} f^2 dv_g \right)^{\frac{1}{2}}.$$

Proof. Remark that if (1.16) holds on Ω, f belongs to $W^{1,2}_{loc}(\Omega)$. As in the Moser iteration technique which we presented above, the goal is to use the Sobolev inequality and to increase the exponent q such that f belongs to $L^q(X)$, but this time only over a ball. Therefore, in order to get the inequality that we are going to iterate, we have to introduce a cut-off function supported on a ball centred at a regular point, and whose gradient decreases in the appropriate way.

Moreover, we need to be sure that we can actually apply iteration on powers of the function f. This means that an inequality of the form (1.16) must hold also for powers f^α, for $\alpha > 1$, possibly with a different constant. We claim that if (1.16) holds on Ω, then for any $\alpha > 1$ we have:

$$\Delta_g(f^\alpha) \leq c\alpha f^\alpha \quad \text{ on } \Omega. \tag{1.17}$$

This also implies that f^α belongs to $W^{1,2}_{loc}(\Omega)$. In order to prove (1.17), for any $\varepsilon > 0$ define $f_\varepsilon = \sqrt{f^2 + \varepsilon^2} > 0$. Consider the Laplacian of f_ε^2 on Ω:

$$f_\varepsilon \Delta_g f_\varepsilon - |df_\varepsilon|^2 = \frac{1}{2}\Delta_g(f_\varepsilon^2) = f\Delta_g f - |df|^2 \leq cf^2 - |df|^2 \leq cf_\varepsilon^2 - |df_\varepsilon|^2.$$

We have then shown that $f_\varepsilon \Delta_g f_\varepsilon \leq cf_\varepsilon^2$ on Ω. For $\alpha > 1$ consider $\Delta_g(f_\varepsilon^\alpha)$. Since x^α is a convex function, non-decreasing on $\mathbb{R}^+$, on Ω we have:

$$\begin{aligned}
\Delta_g(f_\varepsilon^\alpha) &= \alpha(f_\varepsilon^{\alpha-1}\Delta_g f_\varepsilon - (\alpha-1)f_\varepsilon^{\alpha-2}|df_\varepsilon|^2) \\
&\leq \alpha f_\varepsilon^{\alpha-1}\Delta_g f_\varepsilon \\
&\leq c\alpha f_\varepsilon^\alpha.
\end{aligned}$$

where in the last inequality we used the fact that $f_\varepsilon \Delta_g f_\varepsilon \leq cf_\varepsilon^2$ on Ω. Now it suffices to let ε go to zero to obtain (1.17).

Let $R_0 = d_g(x, \Sigma)/2$ and choose $0 < r < R < R_0$. Consider a Lipschitz function φ having compact support in $B(x, R_0)$ such that φ is equal to one in $B(x, r)$, it vanishes outside $B(x, R)$, it takes values between 0 and 1 on $B(x, R)$, and its gradient satisfies:

$$|d\varphi| \leq \frac{2}{(R-r)}$$

Let us consider φf, to which we are going to apply the Sobolev inequality. By a standard formula for the norm in L^2 of the gradient $d(\varphi f)$ we have:

$$\begin{aligned}\int_{B(x,R)} |d(\varphi f)|^2 dv_g &= \int_{B(x,R)} (|d\varphi|^2 f^2 + \varphi^2 f \Delta_g f) dv_g \\ &\leq \int_{B(x,R)} (|d\varphi|^2 f^2 + c\varphi^2 f^2) dv_g \\ &\leq \frac{A_1}{(R-r)^2} \int_{B(x,R)} f^2 dv_g.\end{aligned}$$

for some positive constant A_1. By applying the Sobolev inequality to φf we then obtain:

$$\begin{aligned}\left(\int_{B(x,R)} |\varphi f|^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{n}} &\leq C_s \left(\int_{B(x,R)} \varphi^2 f^2 dv_g + \int_{B(x,R)} |d(\varphi f)|^2 dv_g\right) \\ &\leq C_s \int_{B(x,R)} \varphi^2 f^2 dv_g + \frac{A_1 C_s}{(R-r)^2} \int_{B(x,R)} f^2 dv_g \\ &\leq \frac{A_2}{(R-r)^2} ||f||^2_{L^2(B(x,R))}\end{aligned}$$

If we denote $\gamma = \frac{n}{n-2} > 1$, we have shown that:

$$\|f\|_{L^{2\gamma}(B(x,r))} \leq \left(\frac{A_2}{(R-r)^2}\right)^{\frac{1}{2}} \|f\|_{L^2(B(x,R))} \tag{1.18}$$

We would like to iterate this argument on greater powers of f; since the two radii appearing in the previous inequality are different, we have to define an appropriate sequence of radii in such a way that at the step $j+1$ the radius R_{j+1} coincides withe the previous r_j. Consider for $j \in \mathbb{N}$ the sequence given by:

$$\begin{aligned} r_j &= \left(\frac{1}{2} + 2^{-(j+3)}\right) R_0 \\ R_j &= \left(\frac{1}{2} + 2^{-(j+2)}\right) R_0. \end{aligned}$$

so that we have $R_{j+1} = r_j$ and $(R_j - r_j) = 2^{-j-3} R_0$.

Thanks to (1.17), we are now able to apply the same argument we used for φf to φf^γ, and so on iteratively with γ^j, for $j = 1, \ldots N$. This leads to:

$$||f||_{L^{2\gamma^N}(B(x,r_N))} \leq \prod_{j=0}^{N-1} \left(\frac{2^{2(j+3)} A_2 \gamma^j}{R_0^2}\right)^{\frac{1}{2\gamma^j}} ||f||_{L^2(B(x,3R_0/4))} \tag{1.19}$$

When we let N tend to ∞, the left-hand side converges to the norm in L^∞ of f over the ball $B(x, R_0/2)$, and the first factor in the product in the right-hand side converges to a constant C divided by $R_0^{n/2}$. In fact we can consider its logarithm and we have:

$$\ln\left(\prod_{j=0}^{N-1}\left(\frac{2^{2(j+3)} A_2 \gamma^j}{R_0^2}\right)^{\frac{1}{2\gamma^j}}\right) = \frac{\ln(2)}{2} \sum_{j=0}^{N-1} \frac{j+3}{\gamma^j} + \frac{\ln(\gamma)}{2} \sum_{j=0}^{N-1} \frac{j}{\gamma^j} + \frac{1}{2} \ln\left(\frac{A_2}{R_0^2}\right) \sum_{j=0}^{N-1} \frac{1}{\gamma^j}.$$

Since γ is strictly greater than one, the first two sums converges to a constant as N tends to infinity, while the last one tends to $\dfrac{1}{1-1/\gamma} = \dfrac{n}{2}$. Finally, by passing to the limit as N goes to infinity, we obtain:

$$\|f\|_{L^\infty(B(x,R_0/2))} \leq c_1 \left(\frac{1}{R_0^n} \int_{B(x,3R_0/4)} f^2 dv_g \right)^{\frac{1}{2}}.$$

as we wished. □

The proof of Proposition 1.15 follows easily:

Proof of Proposition 1.15. Let $x \in \Omega$ and $B(x,r)$ a ball of radius $0 < r < d_g(x,\Sigma)/2$, which is entirely contained in Ω. Lemma 1.16 allows us to bound the norm in L^∞ of $|du|$ over a ball $B(x,r/2)$ with the mean of its norm in L^2 over a ball of radius $3r/4$. The square of this last quantity is bounded by some constant times $|\ln(r)|$, thanks to Theorem 1.14. Therefore, we get the desired inequality outside an ε tubular neighbourhood of Σ by choosing an appropriate small radius r. □

Remark 1.17. Since $|du|$ satisfies the estimate (1.15), it is in $L^p(X)$ for any $p \in [1,+\infty)$. In fact, if we denote by m the codimension of the singular set Σ, which is greater or equal to two, we have:

$$\begin{aligned}
\int_X |du|^p dv_g &= \int_{X\setminus\Sigma^\varepsilon} |du|^p dv_g + \int_{\Sigma^\varepsilon} |du|^p dv_g \\
&\leq |\ln(\varepsilon)|^{\frac{p}{2}} \operatorname{Vol}_g(X) + C^p \int_0^\varepsilon \left(\int_{\partial\Sigma^t} |\ln(t)|^{\frac{p}{2}} d\sigma_g \right) dt \\
&\leq |\ln(\varepsilon)|^{\frac{p}{2}} \operatorname{Vol}_g(X) + C_1 \int_0^\varepsilon t^{m-1} |\ln(t)|^{\frac{p}{2}} dt.
\end{aligned}$$

Where we used that the volume of boundary of the tubular neighbourhood of size t is bounded by a constant times the $(m-1)$ power of t. The last integral is clearly finite, therefore $|du|$ belongs to $L^p(X)$.

Remark 1.18. We will show in the next chapters that the hypothesis of Proposition 1.15 hold on a large class of stratified spaces satisfying a lower bound on the Ricci tensor, and for a large class of Schrödinger equations. For example, given a locally Lipschitz function F on $\mathbb{R}$, we will be able to apply this result to an equation of the form

$$\Delta_g u = F(u)$$

This clearly includes the eigenfunctions of the Laplacian.

Moreover, if the metric g is an Einstein metric, we can also apply Proposition 1.15 to the solutions to the Yamabe equation:

$$\Delta_g u + \frac{n(n-2)}{4} u = \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}}.$$

We leave the details for the next chapters, where we define what we mean for Einstein metric and lower Ricci bound in the context of iterated edge metrics, and where this kind of equations will naturally appear.

Chapter 2

Positive Ricci lower bounds on stratified spaces

The aim of this chapter is to study the consequences of a positive lower bound on the Ricci tensor on a stratified space X endowed with an iterated edge metric g. What we mean for Ricci bound on a stratified space is a classical bound for the Ricci tensor on the regular set Ω of X: the iterated edge metric g is a smooth Riemannian metric on Ω, therefore Ric_g is well defined in each regular point, and it makes sense to consider an inequality of the form

$$Ric_g \geq \lambda g \quad \text{on } \Omega, \text{ for } \lambda \in \mathbb{R}.$$

If the equality holds, then we say that g is an Einstein metric and (X, g) an Einstein stratified space. If λ is positive, we say that we have a positive Ricci lower bound. Observe that in this case, we can always rescale the metric in order to get $\lambda = (n-1)$, where n is the dimension of the stratified space.

We also define the class of admissible stratified spaces, which satisfy a positive Ricci lower bound and a second condition on the stratum of minimal codimension equal to two:

Definition 8. An *admissible stratified space* is a stratified space (X^n, g) which satisfies the following assumptions:

(1) If there exists a stratum of codimension 2, its link has diameter smaller than π.

(2) The iterated edge metric g satisfies $Ric_g \geq (n-1)g$ on the dense smooth set Ω.

In this setting, we can extend some well-known results of classical Riemannian geometry: the Lichnerowicz and Obata theorems. The first states that a positive lower bound on the Ricci tensor leads to a lower bound for the first non-zero eigenvalue of the Laplacian, while the second gives a rigidity result in the case that this last lower bound is attained.

At the end of this chapter, we also give a lower bound for the optimal constant appearing in the Sobolev inequality on an admissible stratified space. This implies an upper bound on the diameter which extends the Myers theorem. Both of this results are inspired by a work of D. Bakry and D. with M. Ledoux.

2.1 Tangent cones and Ricci lower bounds

We start by giving some simple consequences of Ricci lower bounds on the tangent cones of a not necessarily admissible stratified space.

Let (X, g) be a stratified space with strata X^j, $j = 1, \dots N$ and links Z_j, each endowed with iterated edge metric k_j. Let us denote by $d_j = n - j - 1$ the dimension of the link Z_j.

A lower bound on the Ricci tensor of g implies that the Ricci tensor associated to the exact cone metric on the tangent cone is non-negative. This leads in turn to a positive Ricci lower bounds for the metrics k_j and the metrics h_x, defined in the previous chapter, on the tangent spheres S_x.

Lemma 2.1. *Let X be a compact stratified space endowed with an iterated edge metric g such that the Ricci tensor is bounded by below. Then for each point $x \in X$ the tangent cone has non-negative Ricci tensor. Furthermore, on each link (Z_j, k_j) we have $Ric_{k_j} \geq (d_j - 1)k_j$.*

Observe that in order to prove this lemma we do not need to assume that the constant λ for which we have $Ric_g \geq \lambda g$ on Ω is positive.

Proof. By definition, the tangent cone at a point x of the stratum X^j is the Gromov-Hausdorff limit of $(X, \varepsilon^{-2}g, x)$ as ε goes to zero. Furthermore, the convergence of the metrics is uniform in C^∞ away from the singular set of X. As a consequence we have:

$$Ric_{g_\varepsilon} = Ric_g \geq \lambda g = \varepsilon^2 \lambda g_\varepsilon \quad \text{on } \Omega.$$

Then when we pass to the limit as ε goes to zero the Ricci tensor of the limit metric $g_{T,x} = dr^2 + r^2 h_x$ must be non-negative. We refer to the formulas for the Ricci tensor of warped products contained in Chapter 3 of [Pet06] in order to deduce that the positivity of the Ricci tensor of $g_{T,x}$ implies:

$$Ric_{h_x} \geq (n-2)h_x$$

Recall that the metric h_x has the form of a doubly warped product:

$$h_x = d\theta^2 + \sin^2\theta g_{\mathbb{S}^{j-1}} + \cos^2\theta k_j.$$

Then again by using the formulas in [Pet06], the metric k_j on the link Z_j must satisfy

$$Ric_{k_j} \geq (d_j - 1)k_j.$$

as we stated above. ∎

We can alternatively assume that a positive Ricci lower bound holds only on the links, and not necessarily on the whole space X. This implies the positivity of the Ricci tensor of the tangent cones.

Lemma 2.2. *Let (Z, k) be a compact stratified space of dimension d with a positive lower bound on the Ricci tensor. Let S be the $(n-d-2)$-fold spherical suspension of Z*

$$S = \left[0, \frac{\pi}{2}\right] \times \mathbb{S}^{n-d-2} \times Z.$$

endowed with the metric:

$$h = d\theta^2 + \sin^2\theta g_{\mathbb{S}^{n-d-2}} + \cos^2\theta k. \tag{2.1}$$

Then the cone $C(S) = (0, \infty) \times S$, endowed with the exact cone metric $g_T = dr^2 + r^2 h$ is a stratified space of dimension n with non-negative Ricci tensor.

Proof. As we observed above, we can assume that the Ricci tensor satisfies the inequality $Ric_k \geq (d-1)k$, up to rescaling by a constant the metric k. By recalling again [Pet06], Chapter 3, page 71, this implies that the Ricci tensor of the metric h is bounded by below as follows:

$$Ric_h \geq (n-2)h.$$

As a consequence the Ricci tensor of $g_T = dr^2 + r^2 h$ on $C(S)$ is non-negative, as we wished. □

2.2 Lichnerowicz Singular Theorem

Before stating our extension of the Lichnerowicz theorem, we give a first example in which we can apply Proposition 1.15. The proof is based on the fact that the classical Bochner-Lichnerowicz formula holds on the regular set Ω: this technique is based on the first part of the proof of Theorem 1.9 in [Bou12].

Proposition 2.3 (Bochner Method)**.** *Let (X, g) be a stratified space with a lower bound on the Ricci tensor. Let F be a locally Lipschitz function on $\mathbb{R}$. Let u be a non negative function in $W^{1,2}(X) \cap L^\infty(X)$ and assume that u is a weak solution to the equation:*

$$\Delta_g u = F(u). \tag{2.2}$$

Then there exists a constant c such that:

$$\Delta_g |du| \leq c|du| \quad on\ \Omega.$$

Proof. For $\varepsilon > 0$, let us introduce

$$f_\varepsilon = \sqrt{|du|^2 + \varepsilon^2} > 0$$

We will consider $\Delta_g(f_\varepsilon^2)$ in order to obtain an inequality of the type $f_\varepsilon \Delta_g f_\varepsilon \leq c f_\varepsilon^2$: dividing by f_ε and letting ε tend to zero will allow us to conclude. We have

$$f_\varepsilon \Delta_g f_\varepsilon - |df_\varepsilon|^2 = \frac{1}{2}\Delta_g(|du|^2 + \varepsilon^2) = (\nabla^* \nabla du, du) - |\nabla du|^2.$$

The Bochner-Lichnerowicz formula holds on the regular set Ω, then by applying it to the equation (2.2) we get:

$$\nabla^* \nabla du + Ric_g(du) = F'(u) du$$

We can now multiply both sides by du. By assumption the function u is bounded and F is locally Lipschitz: then there exists a positive constant c_1 such that on $[-\|u\|_\infty, \|u\|_\infty]$ the derivative of F is bounded by c_1. Moreover, the Ricci tensor Ric_g is bounded by below by λg for some real constant λ, and therefore we obtain:

$$(\nabla^* \nabla du, du) \leq c_1 |du|^2 - \lambda |du|^2.$$

As a consequence there exists a positive constant $c = \max\{c_1 - \lambda, 1\}$ such that:

$$\begin{aligned}(\nabla^* \nabla du, du) - |\nabla du|^2 &\leq c_1 |du|^2 - \lambda |du|^2 - |\nabla du|^2 \\ &\leq c|du|^2 - |\nabla du|^2.\end{aligned}$$

We also observe that, by elementary calculations and Kato's inequality:

$$|df_\varepsilon|^2 = \frac{|du|^2 |\nabla |du||^2}{|du|^2 + \varepsilon^2} \leq |\nabla |du||^2 \leq |\nabla du|^2.$$

and as a consequence we get:

$$\begin{aligned}f_\varepsilon \Delta_g f_\varepsilon - |df_\varepsilon|^2 &= (\nabla^* \nabla du, du) - |\nabla du|^2 \\ &\leq c|du|^2 - |\nabla du|^2 \\ &\leq c f_\varepsilon^2 - |df_\varepsilon|^2.\end{aligned}$$

In conclusion

$$f_\varepsilon \Delta_g f_\varepsilon \leq c f_\varepsilon^2$$

Since f_ε is positive everywhere, we can divide by f_ε and obtain

$$\Delta_g f_\varepsilon \leq c f_\varepsilon$$

By letting ε go to zero, we deduce the desired inequality on $|du|$. $\square$

As a consequence, provided that the hypothesis on the first eigenvalue of the tangent spheres is satisfied as well, we can apply Proposition 1.15 and obtain:

Corollary 2.4. *Let (X^n, g) be a stratified space with a Ricci lower bound. Assume that for any $x \in X$ we have:*

$$\nu_1(S_x) = 1.$$

If u is a solution to the equation (2.2), then for any $\varepsilon > 0$ we have the following estimate of its gradient, away from a tubular neighbourhood Σ^ε of the singular set:

$$\|du\|_{L^\infty(X \setminus \Sigma^\varepsilon)} \leq C\sqrt{|\ln(\varepsilon)|}.$$

We now have all the necessary tools to prove the following:

Theorem 2.5 (Singular Lichnerowicz Theorem)**.** *Let (X, g) be an admissible stratified space of dimension n. Any non-zero eigenvalue λ of the Laplacian Δ_g is greater than or equal to n.*

Remark 2.6. In [BS14] the authors give an analogous Lichnerowicz theorem for spherical cones $\Sigma(M)$ (considered as metric measure spaces) on a compact Riemannian manifold (M, g) with lower Ricci bound $Ric_g \geq (n-1)g$. They use the existence of a curvature dimension condition $CD(n, n+1)$ on $\Sigma(M)$ in the generalized sense of Sturm and Lott-Villani. Our theorem applies more generally to cones over any stratified space (X, g) having a positive lower Ricci bound on the regular set Ω.

Proof. We proceed by iteration on the dimension n of the space.

If $n = 1$, by our hypothesis X must be a circle of diameter smaller than π. As a consequence, the first eigenvalue of the Laplacian is greater than 1, and the theorem holds in one dimension.

Assume that the statement is true for any dimension until $(n-1)$ and consider an admissible stratified space X of dimension n. For any $x \in X$, the tangent sphere S_x is an admissible stratified space of dimension $(n-1)$. In fact, by Lemma 2.1, the condition $Ric_g \geq (n-1)g$ implies that for any x the metric h_x satisfies $Ric_{h_x} \geq (n-2)h_x$.

Therefore, by the iteration argument, for any $x \in X$:

$$\lambda_1(S_x) \geq (n-1)$$

Then the stratified space (X^n, g) satisfies the hypothesis of Corollary 2.4 and we can apply this result to any eigenfunction φ of the Laplacian:

$$\Delta_g \varphi = \lambda \varphi. \tag{2.3}$$

Therefore for any $\varepsilon > 0$ we have:

$$\|d\varphi\|_{L^\infty(X \setminus \Sigma^\varepsilon)} \leq C\sqrt{|\ln(\varepsilon)|}.$$

Since we have an estimation of the behaviour of $d\varphi$ depending on the distance from the singular set, the rest of the proof is an adaptation of the classical one by means of a well-chosen family of cut-off functions. Consider for $\varepsilon > 0$ a cut-off function ρ_ε, being equal to one outside Σ^ε, vanishing on some smaller tubular neighbourhood of Σ and such that between the two tubular neighbourhoods ρ_ε takes values between 0 and 1. We are going to specify the choice of such function in the following.

We proceed here like in the setting of smooth compact manifolds: by the Bochner-Lichnerowicz formula applied to equation (2.3), we have on the regular set Ω:

$$\nabla^*\nabla d\varphi + Ric_g(d\varphi) = \lambda d\varphi$$

We then consider the Laplacian of $|d\varphi|^2$ and get:

$$\frac{1}{2}\Delta_g|d\varphi|^2 = (\nabla^*\nabla d\varphi, d\varphi) - |\nabla d\varphi|^2 \leq \lambda|d\varphi|^2 - (n-1)|d\varphi|^2 - |\nabla d\varphi|^2. \tag{2.4}$$

If we multiply (2.4) by ρ_ε and integrate by parts we obtain:

$$\int_X \Delta_g(\rho_\varepsilon)\frac{|d\varphi|^2}{2}dv_g \leq \int_X \rho_\varepsilon((\lambda-(n-1))|d\varphi|^2 - |\nabla d\varphi|^2)dv_g \tag{2.5}$$

We study the right-hand side and we consider the first term. By elementary calculations and integration by parts formula we can rewrite:

$$\begin{aligned}\int_X \rho_\varepsilon|d\varphi|^2 dv_g &= \int_X (d(\rho_\varepsilon\varphi), d\varphi) - \varphi(d\rho_\varepsilon, d\varphi))dv_g \\ &= \int_X \rho_\varepsilon\varphi\Delta_g\varphi dv_g - \int_X \varphi(d\rho_\varepsilon, d\varphi)dv_g \\ &= \frac{1}{\lambda}\int_X \rho_\varepsilon(\Delta_g\varphi)^2 dv_g - \int_X \varphi(d\rho_\varepsilon, d\varphi)dv_g.\end{aligned}$$

Therefore, by going back to (2.5), we get:

$$\int_X \Delta_g(\rho_\varepsilon)\frac{|d\varphi|^2}{2}dv_g \leq \int_X \rho_\varepsilon\left(\left(1-\frac{n-1}{\lambda}\right)(\Delta_g\varphi)^2 - |\nabla d\varphi|^2\right)dv_g + \lambda\int_X \varphi(d\varphi, d\rho_\varepsilon)dv_g. \tag{2.6}$$

In order conclude the proof, we need to choose a family of cut-off functions ρ_ε such that when ε goes to zero we have:

(i) the left-hand side of (2.6) tends to zero;

(ii) the last term of the right-hand side in (2.6) tends to zero.

If we can find a family of cut-off functions satisfying these two conditions, then when we pass to the limit as ε goes to zero we obtain:

$$\left(1-\frac{(n-1)}{\lambda}\right)\int_X (\Delta_g\varphi)^2 dv_g - \int_X |\nabla d\varphi|^2 dv_g \geq 0$$

Moreover, by Cauchy-Schwarz inequality we have:

$$|\nabla du|^2 \geq \frac{(\Delta_g\varphi)^2}{n}$$

As a consequence, we finally have:

$$\left(1 - \frac{(n-1)}{\lambda} - \frac{1}{n}\right) \int_X (\Delta_g \varphi)^2 dv_g \geq 0.$$

which clearly leads to $\lambda \geq n$.
It remains to show that it is actually possible to construct a family of cut-off functions having the properties (i) and (ii). This is done in the following.

Choice of the family of cut-off functions

We have to distinguish two different cases, whether the codimension m of Σ is strictly greater than two, or equal to two.

Case 1: Firstly assume that $m > 2$. Consider $\varepsilon > 0$ and the tubular neighbourhoods Σ^ε of size ε and $\Sigma^{2\varepsilon}$ of size 2ε. We want to build a cut-off function ρ_ε which is equal to one away from $\Sigma^{2\varepsilon}$ and vanishes on Σ^ε. Moreover, we need the gradient $d\rho_\varepsilon$ and the Laplacian $\Delta_g\rho_\varepsilon$ to decay "fast enough" as ε tends to zero. We will obtain ρ_ε from a harmonic function, as explained in the following.
Let h_ε be the harmonic extension of the function which is equal to 1 on the boundary of $\Sigma^{2\varepsilon}$ and vanishes on the boundary of Σ^ε, i.e. h_ε satisfies:

$$\begin{cases} \Delta_g h_\varepsilon = 0 \\ h_\varepsilon = 1 \text{ on } \partial\Sigma^{2\varepsilon} \\ h_\varepsilon = 0 \text{ on } \partial\Sigma^\varepsilon. \end{cases}$$

The harmonic extension has a variational characterization: if we consider the Dirichlet energy $\mathcal{E}$ defined by:

$$\mathcal{E}(\varphi) = \int_{\Sigma^{2\varepsilon}\setminus\Sigma^\varepsilon} |d\varphi|^2 dv_g.$$

then h_ε attains the infimum of the functional $\mathcal{E}$ among all functions $\varphi \in W^{1,2}(X)$ taking values 1 on $\partial\Sigma^{2\varepsilon}$ and vanishing on $\partial\Sigma^\varepsilon$.
Let r be the distance function from the singular set Σ, i.e. $r(x) = d_g(x, \Sigma)$, and consider the function g_ε that we introduced in the first chapter:

$$g_\varepsilon(r) = \begin{cases} 1 \text{ on } X \setminus \Sigma^{2\varepsilon} \\ \dfrac{r}{\varepsilon} - 1 \text{ on } \Sigma^{2\varepsilon} \setminus \Sigma^\varepsilon \\ 0 \text{ on } \Sigma^\varepsilon. \end{cases} \qquad |d\psi_\varepsilon| = \frac{1}{\varepsilon}.$$

We have already shown that there exists a positive constant B such that the Dirichlet energy of g_ε is bounded by above:

$$\mathcal{E}(g_\varepsilon) \leq B\varepsilon^{m-2}.$$

Therefore, by the variational characterization of h_ε, its Dirichlet energy is smaller than the one of g_ε, $\mathcal{E}(h_\varepsilon) \leq \mathcal{E}(g_\varepsilon)$, and we obtain:

$$\mathcal{E}(h_\varepsilon) \leq B\varepsilon^{m-2}. \tag{2.7}$$

However, h_ε is not necessarily smooth. The cut-off function ρ_ε will be obtained by composing h_ε with a smooth function ρ vanishing on $(-\infty, \frac{1}{4}]$ end being equal to one on $[\frac{3}{4}, +\infty)$: more precisely, $\rho_\varepsilon = \rho \circ h_\varepsilon$. As a consequence we have:

$$d\rho_\varepsilon = (\rho' \circ h_\varepsilon)dh_\varepsilon \quad \text{and} \quad \Delta_g\rho_\varepsilon = -(\rho'' \circ h_\varepsilon)|dh_\varepsilon|^2.$$

Since ρ is smooth and chosen independently from ε, there exist two constants c_1, c_2, not depending on ε, such that:

$$|d\rho_\varepsilon| \leq c_1 |dh_\varepsilon|, \quad \text{and} \quad |\Delta \rho_\varepsilon| \leq c_2 |dh_\varepsilon|^2.$$

We claim that our choice of ρ_ε satisfies (i) and (ii). For what concerns the first condition, by using (2.2) on the gradient $d\varphi$ and (2.7), we obtain:

$$\int_{\Sigma^{2\varepsilon} \setminus \Sigma^\varepsilon} |\Delta_g \rho_\varepsilon| |d\varphi|^2 dv_g \leq c_2 \int_{\Sigma^{2\varepsilon} \setminus \Sigma^\varepsilon} |dh_\varepsilon|^2 |d\varphi|^2 dv_g \leq C_2 |\ln(\varepsilon)| \varepsilon^{m-2}.$$

which tends to zero as ε goes to zero. As for the second condition (ii), by using Cauchy-Schwarz inequality twice and the estimate we have on $|d\rho_\varepsilon|$, we get:

$$\begin{aligned}
\int_{\Sigma^{2\varepsilon} \setminus \Sigma^\varepsilon} (d\rho_\varepsilon, d\varphi) dv_g &\leq \int_{\Sigma^{2\varepsilon} \setminus \Sigma^\varepsilon} |d\rho_\varepsilon| |d\varphi| dv_g \\
&\leq \left(\int_{\Sigma^{2\varepsilon} \setminus \Sigma^\varepsilon} |d\rho_\varepsilon|^2 dv_g \right)^{\frac{1}{2}} \left(\int_{\Sigma^{2\varepsilon} \setminus \Sigma^\varepsilon} |d\varphi|^2 dv_g \right)^{\frac{1}{2}} \\
&\leq c_1' \varepsilon^{\frac{m}{2}} \sqrt{|\ln(\varepsilon)|} \left(\int_{\Sigma^{2\varepsilon} \setminus \Sigma^\varepsilon} |dh_\varepsilon|^2 dv_g \right)^{\frac{1}{2}} \\
&\leq c_1'' \varepsilon^{m-1} \sqrt{|\ln(\varepsilon)|}.
\end{aligned}$$

which also tends to zero with ε.

<u>**Case 2**</u>: Consider $m = 2$. The cut-off function ρ_ε will be equal to one outside Σ^ε and it will vanish in Σ^{ε^2}, for $0 < \varepsilon < 1$. In this case too ρ_ε is obtained by "smoothing" the harmonic function h_ε being equal to one on $\partial\Sigma^\varepsilon$ and vanishing on $\partial\Sigma^{\varepsilon^2}$. We will be able to show that the Dirichlet energy of h_ε tends to zero when ε goes to zero as $|\ln(\varepsilon)|^{-1}$. A priori this estimate does not suffices to show (i) and (ii), but only implies that the two integrals are bounded. For this reason we will need to give a more detailed study: we are going to prove that in fact $|d\varphi|$ belongs to $W^{1,2}(X) \cap L^\infty(X)$.

Let h_ε be the harmonic function solving:

$$\begin{cases} \Delta_g h_\varepsilon = 0 \\ h_\varepsilon = 1 \text{ on } \partial\Sigma^\varepsilon \\ h_\varepsilon = 0 \text{ on } \partial\Sigma^{\varepsilon^2}. \end{cases}$$

We can found a test function f_ε such that the Dirichlet energy $\mathcal{E}(f_\varepsilon)$ is bounded by a constant times $|\ln(\varepsilon)|^{-1}$. We consider the function f_ε defined in Lemma 1.6:

$$f_\varepsilon(r) = \begin{cases} 1 \quad \text{on } X \setminus \Sigma^\varepsilon \\ \left(2 - \dfrac{\ln(r)}{\ln(\varepsilon)}\right) \quad \text{on } \Sigma^\varepsilon \setminus \Sigma^{\varepsilon^2} \\ 0 \quad \text{on } \Sigma^{\varepsilon^2}. \end{cases} \qquad |df_\varepsilon| = \frac{1}{r|\ln(\varepsilon)|}.$$

We have already proven that there exists a positive constant A, independent of ε, for which we have:

$$\left(\int_{\Sigma^{\varepsilon}\setminus\Sigma^{\varepsilon^2}} |df_{\varepsilon}|^2 dv_g\right) \leq \frac{A}{|\ln(\varepsilon)|}. \tag{2.8}$$

Then by the variational characterization of h_{ε} we obtain:

$$\mathcal{E}(h_{\varepsilon}) \leq \mathcal{E}(f_{\varepsilon}) \leq \frac{A}{|\ln(\varepsilon)|}.$$

Furthermore, thanks to our estimate on the behaviour of $d\varphi$ we obtain:

$$\begin{aligned}
\int_{\Sigma^{\varepsilon}\setminus\Sigma^{\varepsilon^2}} |dh_{\varepsilon}|^2|d\varphi|^2 dv_g &\leq C|\ln(\varepsilon^2)| \int_{\Sigma^{\varepsilon}\setminus\Sigma^{\varepsilon^2}} |dh_{\varepsilon}|^2 dv_g \\
&\leq 2C^2|\ln(\varepsilon)| \int_{\Sigma^{\varepsilon}\setminus\Sigma^{\varepsilon^2}} |df_{\varepsilon}|^2 dv_g \\
&\leq 2C^2|\ln(\varepsilon)|\frac{A}{|\ln(\varepsilon)|} \leq D.
\end{aligned}$$

where D is a positive constant independent of ε.

If we replace ρ_{ε} by f_{ε} in (2.5) and we let ε go to zero we then obtain a finite term B_1 on the left-hand side. Therefore we obtain the following estimate:

$$\int_X |\nabla d\varphi|^2 dv_g + B_1 \leq \int_X (\lambda - (n-1))|d\varphi|^2 dv_g.$$

Recall that φ belongs to $W^{1,2}(X)$, so that the norm of $|d\varphi|$ in $L^2(X)$ is finite. Then the previous inequality tells us that also $|\nabla d\varphi|$ must be in $L^2(X)$, and so $\nabla|d\varphi|$ too, since we have clearly $|\nabla|d\varphi|| \leq |\nabla d\varphi|$. As a consequence we have that $|d\varphi|$ belongs to $W^{1,2}(X)$. This allows us to get more regularity on $|d\varphi|$, by using the fact that on Ω the gradient satisfies $\Delta_g|d\varphi| \leq c|d\varphi|$.

Claim: The gradient $d\varphi$ belongs to $L^{\infty}(X)$.

Proof. Let us call $u = |d\varphi|$ for simplicity. We state that u satisfies the weak inequality

$$\Delta_g u \leq cu. \tag{2.9}$$

on the whole X. This means that for any $\psi \in W^{1,2}(X)$, $\psi \geq 0$ we have:

$$\int_X (du, d\psi)_g dv_g \leq c \int_X u\psi dv_g. \tag{2.10}$$

We already proved that $\Delta_g u \leq cu$ strongly on Ω, then we know that for any $\psi \in W^{1,2}(X)$ we have:

$$\int_{\Omega} \psi\Delta_g u dv_g \leq c \int_{\Omega} u\psi dv_g.$$

In order to extend this inequality to the whole X and obtain (2.9), we consider f_ε defined as above, $0 \leq f_\varepsilon \leq 1$ and we replace ψ by $f_\varepsilon\psi$. By integrating by parts we obtain:

$$\int_X (d(f_\varepsilon\psi), du)_g dv_g \leq c \int_X f_\varepsilon \psi u + \int_X \psi(df_\varepsilon, du)_g dv_g. \tag{2.11}$$

We can use Cauchy-Schwarz inequality twice on the second term and obtain:

$$\begin{aligned}\int_X \psi(df_\varepsilon, du)_g dv_g &\leq B_2 \|du\|_2 \left(\int_X |df_\varepsilon|^2 dv_g\right)^{\frac{1}{2}} \\ &\leq B_3 \|du\|_2 \frac{1}{\sqrt{|\ln(\varepsilon)|}}.\end{aligned}$$

Where we used the estimate (2.8) on the gradient of f_ε. Since the norm in $L^2(X)$ of the Hessian $du = \nabla d\varphi$ is finite, the second term in (2.11) tends to zero when ε goes to zero. Then letting ε go to zero in (2.11) implies (2.10), as we wished. Since (2.9) is proven, Moser's iteration technique in Proposition 1.8 of [ACM14] assures that $|d\varphi| \in L^\infty(X)$. □

We are finally in the position to show that in codimension $m = 2$ a family of cut-off functions satisfying (i) and (ii) exists: define $\rho_\varepsilon = \rho \circ h_\varepsilon$ for the same smooth function ρ as before. We have for c_1, c_2 independent of ε

$$|d\rho_\varepsilon| \leq c_1 |dh_\varepsilon| \quad |\Delta_g \rho_\varepsilon| \leq c_2 |dh_\varepsilon|^2.$$

The estimate on the Dirichlet energy on h_ε and the fact that the norm of the gradient $|d\varphi|$ in $L^\infty(X)$ is finite assure that ρ_ε is the desired cut-off function. For the condition (i) we obtain:

$$\int_X |\Delta_g \rho_\varepsilon||d\varphi| dv_g \leq c_2' \int_X |dh_\varepsilon|^2 dv_g \leq \frac{c_2' A}{|\ln(\varepsilon)|}.$$

which tends to zero as ε goes to zero. For the condition (ii) we use Cauchy-Schwarz inequality twice and we get:

$$\begin{aligned}\int_X (d\rho_\varepsilon, d\varphi)_g dv_g &\leq \left(\int_{\Sigma^{\varepsilon^2} \setminus \Sigma^\varepsilon} |d\rho_\varepsilon|^2 dv_g\right)^{\frac{1}{2}} \left(\int_{\Sigma^{\varepsilon^2} \setminus \Sigma^\varepsilon} |d\varphi|^2\right)^{\frac{1}{2}} \\ &\leq c_1' \varepsilon \left(\int_X |dh_\varepsilon|^2 dv_g\right)^{\frac{1}{2}} \\ &\leq \frac{c_1'' \varepsilon}{\sqrt{|\ln(\varepsilon)|}}.\end{aligned}$$

which tends to zero as ε goes to zero.

We have found an appropriate family of cut-off functions for any codimension of the singular set Σ: this concludes the proof of the theorem. □

Remark 2.7. Observe that we have proven that the gradient of any eigenfunction φ of the Laplacian is such that $|d\varphi|$ belongs to $W^{1,2}(X) \cap L^{\infty}(X)$ not only in the case of codimension $m = 2$, but, using the same argument, also for $m > 2$.

Remark 2.8. In the discussion above for the choice of the cut-off function in codimension $m > 2$ (respectively $m = 2$), we obtain ρ_ε by smoothing a harmonic function h_ε and by considering $\rho \circ \psi_\varepsilon$ (respectively $\rho \circ f_\varepsilon$) because we need a condition on the Laplacian of ρ_ε. The distance function from Σ is not necessarily smooth: we know that almost everywhere $|dr|^2 = 1$, but we do not have information on the behaviour of its Laplacian.

Remark 2.9. By our definition of admissible stratified space, we are excluding the existence of a stratum of codimension 2 whose link is a circle $\mathbb{S}^1_a$, of radius a bigger or equal to one. Recall that the classical Lichnerowicz theorem does not hold for $\mathbb{S}^1_a$, since the first eigenvalue of the Laplacian is equal to $1/a^2 < 1$: the first iterative step in our proof could not be applied.

We can state a consequence of the previous Theorem and of Lemma 2.1 on the spectral geometry of the links of a stratified space:

Lemma 2.10. *Let (X, g) be a stratified space which satisfies a Ricci lower bound, not necessarily positive, and which does not posses any strata of codimension two and cone angle larger than 2π. Then for any x in X we have $\nu_1(S_x) = 1$, and in particular $\nu(X)$ is equal to* 1.

In particular, under the assumptions of the previous lemma, the hypothesis of Corollary 2.4 is always satisfied. We can then reformulate it in the following way:

Corollary 2.11. *Let (X, g) be a stratified space of dimension n. Assume that it has no codimension 2 stratum with link of diameter larger than π and that the Ricci tensor of g is bounded by below. Let F be a locally Lipschitz function on $\mathbb{R}$ and $u \in W^{1,2}(X) \cap L^{\infty}(X)$ a non-negative solution to $\Delta_g u = F(u)$. Then for any $\varepsilon > 0$ we have:*

$$\|du\|_{L^{\infty}(X \setminus \Sigma^{\varepsilon})} \leq C\sqrt{|\ln(\varepsilon)|}.$$

If we use the Bochner-Lichnerowicz formula, integration by parts with the same cut-off functions as in the previous proof, and the argument of Claim 1, we can also get a better regularity result:

Corollary 2.12. *Let (X, g) be a stratified space of dimension n. Assume that it has no codimension 2 stratum with link of diameter larger than π and that the Ricci tensor of g is bounded by below. Let F be a locally Lipschitz function on $\mathbb{R}$ and $u \in W^{1,2}(X) \cap L^{\infty}(X)$ a non-negative solution to $\Delta_g u = F(u)$. Then the gradient du is such that $|du|$ belongs to $W^{1,2}(X) \cap L^{\infty}(X)$.*

2.3 An estimate of Sobolev best constant

The following theorem is inspired by a result of D. Bakry contained in [Bak94] (and also proven by S. Ilias [Ili83]): given a smooth compact Riemannian manifold (M^n, g) with a Ricci positive lower bound of the form $Ric_g \geq (n-1)g$, it is possible to give an explicit estimate of the best constant appearing in the Sobolev inequality. Such estimate depends only on the dimension and on the volume of M with respect to g.

Thanks to the Lichnerowicz theorem 2.5, we can prove an analogous result for an admissible stratified space.

Theorem 2.13. *Let X be an admissible stratified space of dimension n. Then for any $1 < p \leq 2n/(n-2)$ a Sobolev inequality of the following form holds:*

$$V^{1-\frac{2}{p}} \|f\|_p^2 \leq \|f\|_2^2 + \frac{p-2}{n} \|df\|_2^2 . \tag{2.12}$$

where V is the volume of X with respect to the metric g.

Proof. All along this proof we are going to use the renormalized measure $d\mu = V^{-1} dv_g$, where $V = Vol_g(X)$. By Theorem 2.5, we know that the first non-zero eigenvalue of the Laplacian is greater than the dimension n; moreover, as we recalled in the first Chapter, the Sobolev's inequality holds on X (see Proposition 2.2 in [ACM14]). The lower bound on the spectrum of the Laplacian, the Sobolev's inequality and Lemma 4.1 in [Bak94] imply that there exists a positive constant γ such that

$$\|f\|_{\frac{2n}{n-2}}^2 \leq \|f\|_2^2 + \gamma \|df\|_2^2 .$$

where all the norms from now on are with respect to $L^p(X, d\mu)$. By using interpolation between 2 and $\frac{2n}{n-2}$, it is easy to see that for any $p < \frac{2n}{n-2}$ and for any $\delta > 0$ we have the following inequality:

$$\|f\|_p^2 \leq (1+\delta) \|f\|_2^2 + \gamma_0 \|df\|_2^2$$

We denote by γ_0 the best constant appearing in the previous inequality. We are going to show that γ_0 is smaller then $(p-2)/n$, for any choice of $\delta > 0$. By coming back to the measure dv_g, we will get the power $1 - 2/p$ of the volume and therefore the inequality (2.12) will hold on X.

Consider a minimizing sequence for γ_0, i.e. a sequence of positive functions $(f_n)_n$ in $W^{1,2}(X)$ such that the quotient

$$\frac{\|f_n\|_p^2 - (1+\delta) \|f_n\|_2^2}{\|df_n\|_2^2}$$

converges to γ_0. We can assume without loss of generality that $||f_n||_2 = 1$. Then $(f_n)_n$ is bounded in $L^p(X)$ and by the compact embedding of $W^{1,2}(X)$ in $L^p(X)$ we can deduce that there exists a positive function f in $W^{1,2}(X)$ such that $(f_n)_n$ converges weakly to f in $W^{1,2}(X)$, and strongly in $L^p(X)$. Thanks to the normalization of the norms in

$L^2(X)$ of f_n, f is not vanishing everywhere, and thanks to the the choice of $\delta > 0$, f cannot be constant. Moreover, it satisfies the following equation on X:

$$\gamma_0 \Delta_g f + (1+\delta) f = A f^{p-1}. \tag{2.13}$$

where $A = \|f\|_p^{2-p}$ is a finite constant. As we observed in the first chapter, we can apply to the equation (2.13) the Moser iteration technique (see Remark 1.13), as in Proposition 1.8 in [ACM14], in order to show that f is bounded. Since the Ricci tensor is bounded by below and $f(x) = (Ax^{p-1} - (1+\delta)x)$ is a locally Lipschitz function, we can apply Corollary 2.12: then, the gradient of f is bounded and belongs to $W^{1,2}(X)$.

We can express f as the power of a function u, i.e. $f = u^\alpha$ for some α that will be chosen later. Then u is also positive, bounded and its gradient satisfies the same estimate as $|df|$ away from a neighbourhood of the singular set Σ.
We can rewrite (2.13) in the form:

$$Au^{\alpha(p-2)} = (1+\delta) + \gamma_0 \frac{\Delta_g(u^\alpha)}{u^\alpha} = (1+\delta) + \alpha\gamma_0 \left(\frac{\Delta_g u}{u} - (\alpha - 1)\frac{|du|^2}{u^2} \right) \tag{2.14}$$

D. Bakry's proof consists in multiplying this equation for an appropriate factor, and then by integrating it. He finds a factor depending on γ_0^{-1}, p and n, multiplies by the L^2-norm of du, and he bounds it by below by some quantity, which is positive when α is well-chosen. We will proceed in a similar way, by taking care of introducing a cut-off function, because we are allowed to use the equation (2.14) and integration by parts only on the regular set Ω.

Fix $\varepsilon > 0$ and consider the cut-off function ρ_ε chosen in the proof of Theorem 2.5 and depending on the minimal codimension of the singular set. We multiply (2.14) by $\rho_\varepsilon u \Delta_g u$ and integrate on X:

$$\begin{aligned} A \int_X \rho_\varepsilon u^{1+\alpha(p-2)} \Delta_g u d\mu &= (1+\delta) \int_X \rho_\varepsilon u \Delta_g u \\ &+ \gamma_0 \alpha \left(\int_X \rho_\varepsilon (\Delta_g u)^2 d\mu - (\alpha-1) \int_X \rho_\varepsilon \frac{\Delta_g u}{u} |du|^2 d\mu \right). \end{aligned} \tag{2.15}$$

When integrating by parts the left-hand side we obtain:

$$\begin{aligned} A \int_X \rho_\varepsilon u^{1+\alpha(p-2)} \Delta_g u d\mu &= \int_X u^{1+\alpha(p-2)} (d\rho_\varepsilon, du)_g d\mu \\ &+ (1+\alpha(p-2)) \int_X \rho_\varepsilon u^{\alpha(p-2)} |du|^2 d\mu. \end{aligned}$$

Since u is positive and bounded, we can bound $u^{1+\alpha(p-2)}$ by a positive constant independent of ε. Then first term, which contains $(du, d\rho_\varepsilon)_g$, tends to zero as ε goes to zero as we have shown in the proof of Theorem 2.5. In the second term we will replace $u^{\alpha(p-2)}$ by its value given by (2.14), and this allows one to simplify the constant A, which will not appear in the following.
As for the right-hand side of (2.15), consider the first term:

$$\int_X \rho_\varepsilon (u \Delta_g u) d\mu = \int_X u (du, d\rho_\varepsilon)_g d\mu + \int_X \rho_\varepsilon |du|^2 d\mu.$$

and when we let ε tends to zero, since as before u is bounded, we simply get the L^2-norm of du, both for the case $m > 2$ and $m = 2$.
Therefore, after some elementary computation we obtain:

$$\begin{aligned} \frac{1+\delta}{\gamma_0}(p-2)\int_X \rho_\varepsilon |du|^2 d\mu = \int_X \rho_\varepsilon (\Delta_g u)^2 d\mu \\ + (\alpha-1)(1+\alpha(p-2))\int_X \rho_\varepsilon \frac{|du|^4}{u^2} d\mu \\ - \alpha(p-1)\int_X \rho_\varepsilon \frac{\Delta_g u}{u}|du|^2 d\mu + o(1). \end{aligned} \tag{2.16}$$

where we replaced the two terms containing du and $d\rho_\varepsilon$ by a term $o(1)$ which tends to zero as ε goes to zero. Let us denote:

$$I_1 = \int_X \rho_\varepsilon (\Delta_g u)^2 dv_g.$$
$$I_2 = \int_X \rho_\varepsilon \frac{\Delta_g u}{u}|du|^2 dv_g.$$

We are going to bound by below I_1 by integrating the Bochner-Lichnerowicz formula, which holds on the regular set Ω, and to give an alternative expression for I_2 by integrating by parts.
Consider firstly I_1. We multiply the Bochner-Lichnerowicz formula

$$(du, d\Delta_g u)_g = \Delta_g \frac{|du|^2}{2} + |\nabla du|^2 + Ric_g(du, du) \text{ on } \Omega.$$

by the cut-off function ρ_ε and integrate. Recall that by hypothesis we have $Ric_g \geq (n-1)g$.
By rewriting $\rho_\varepsilon(du, d\Delta_g u)_g = (du, d(\rho_\varepsilon \Delta_g u))_g - \Delta_g u(du, d\rho_\varepsilon)_g$ and integrating by parts, we then obtain:

$$\begin{aligned} \int_X \rho_\varepsilon (\Delta_g u)^2 d\mu \geq \int_x \rho_\varepsilon (|\nabla du|^2 dv_g + (n-1)|du|^2) d\mu \\ + \int_X \Delta_g \rho_\varepsilon \frac{|du|^2}{2} d\mu + \int_X \Delta_g u (du, d\rho_\varepsilon)_g d\mu. \end{aligned} \tag{2.17}$$

Remark that thanks to (2.14), and the fact that u is bounded, we know that $\Delta_g u$ can be split in the sum of a bounded term and a second term depending on $|du|^2$: it is equal to

$$\Delta_g u = \frac{1}{\alpha} u \left(\frac{\Delta_g f}{u^\alpha} + \alpha(\alpha-1)\frac{|du|^2}{u^2} \right).$$

We know that u is strictly positive and bounded, then the same is true for u^{-1}, and $\Delta_g f$ is bounded too. Furthermore, we observed at the end of the first chapter, Remark 1.17, that $|du|$ belongs to $L^p(X)$, for all $p \in [1; +\infty)$. As a consequence $\Delta_g u$ also belongs to $L^p(X)$ for $p \in [1; +\infty)$. Then we can bound the last term in (2.17) by using Cauchy-Schwarz inequality:

$$\int_X \rho_\varepsilon \Delta_g u (du, d\rho_\varepsilon)_g d\mu \leq \left(\int_X (\rho_\varepsilon \Delta_g u)^2 d\mu \right)^{\frac{1}{2}} \left(\int_X (du, d\rho_\varepsilon)_g^2 d\mu \right)^{\frac{1}{2}}.$$

where the first factor is finite, and the second one tends to zero as ε goes to zero. Then the last term in (2.17) tends to zero as ε goes to zero. As for the term containing $\Delta_g \rho_\varepsilon$, we know form the proof of Theorem 2.5 that this tends to zero as ε goes to zero.

We can modify (2.17) a bit more. We decompose the Hessian ∇du in its traceless part A plus $-(\Delta_g u/n)g$, since $\Delta_g u = -\mathrm{tr}(\nabla du)$. Then the square norm of ∇du is equal to $|A|^2 + (\Delta_g u)^2/n$, and therefore we get:

$$\int_X \rho_\varepsilon (\Delta_g u)^2 d\mu \geq \frac{n}{n-1} \int_X \rho_\varepsilon |A|^2 d\mu + n \int_X \rho_\varepsilon |du|^2 d\mu + o(1). \tag{2.18}$$

This will be the appropriate bound by below for I_1.
Now consider I_2 and integrate by parts:

$$I_2 = 2 \int_X \rho_\varepsilon \frac{\nabla du(du, du)}{u} d\mu - \int_X \rho_\varepsilon \frac{|du|^4}{u^2} d\mu + \int_X \frac{|du|^2}{u} (d\rho_\varepsilon, du)_g d\mu.$$

With the same observations as before ($|du| \in L^p(X)$ for all $p \in [1 + \infty)$ and Cauchy-Schwarz inequality), we can say that the last term in this expression tends to zero as ε goes to zero. We can decompose again the Hessian ∇du in $\nabla du = A - \frac{\Delta_g u}{n} g$. As a consequence we can write:

$$I_2 = \frac{2n}{n+2} \int_X \rho_\varepsilon \frac{A(du, du)}{u} d\mu - \frac{n}{n+2} \int_X \rho_\varepsilon \frac{|du|^4}{u^2} d\mu + o(1). \tag{2.19}$$

We can now replace this expression for I_2 and the bound by below (2.18) for I_1 in (2.16); after passing to the limit as ε and δ go to zero we obtain:

$$\begin{aligned} \left(\frac{1}{\gamma_0}(p-2) - n \right) \int_X |du|^2 d\mu \geq & \frac{n}{n-1} \int_X |A|^2 d\mu \\ & - \alpha(p-1) \frac{2n}{n+2} \int_X \frac{A(du, du)}{u} d\mu \\ & + C(\alpha) \int_X \frac{|du|^4}{u^2} d\mu. \end{aligned} \tag{2.20}$$

where

$$C(\alpha) = (\alpha - 1)(1 + \alpha(p-2)) + \alpha(p-1) \frac{n}{n+2}.$$

The first two terms in the left-hand side of (2.20) can be interpreted as a part of a square norm for some convenient coefficient: we can rewrite in fact

$$\begin{aligned} \left(\frac{1}{\gamma_0}(p-2) - n \right) \int_X |du|^2 d\mu \geq & \frac{n}{n-1} \left(\int_X \left| A + \beta \frac{du \otimes du}{u} \right|^2 d\mu \right) \\ & + \left(C(\alpha) - \beta^2 \frac{n}{n-1} \right) \int_X \frac{|du|^4}{u^2} d\mu. \end{aligned}$$

where we have chosen:

$$\beta = -\alpha(p-1)\frac{n-1}{n+2}$$

We denote by $T = \frac{du \otimes du}{u}$. Then, recalling that A is traceless, we have

$$|A+\beta T|^2 \geq \frac{1}{n}\mathrm{tr}(A+\beta T)^2 = \frac{\beta^2}{n}\frac{|du|^4}{u^2}.$$

Replacing this in the previous inequality, we finally get:

$$\left(\frac{1}{\gamma_0}(p-2)-n\right)\int_X |du|^2 d\mu \geq (C(\alpha)-\beta^2)\int_X \frac{|du|^4}{u^2}d\mu. \tag{2.21}$$

We remark that $C(\alpha)-\beta^2$ is a quadratic expression in α. Its discriminant equals:

$$-\frac{4n(p-1)((n-2)p-2n)}{(n+2)^2}$$

which is positive for $1 < p < \frac{2n}{n-2}$. Therefore, thanks to our hypothesis, we can choose α in such a way that the right-hand side of (2.21) is a positive quantity. As a consequence we get for any $1 < p < \frac{2n}{n-2}$:

$$\frac{1}{\gamma_0} \geq \frac{n}{p-2}.$$

which gives the desired Sobolev inequality. We can pass to the limit as p tends to $\frac{2n}{n-2}$ and get the result for $\frac{2n}{n-2}$ as well. $\square$

A consequence of Sobolev inequality: singular Myers theorem

Another classical result holding for smooth Riemannian manifolds is the Myers theorem: if (M^n, g) is complete, connected, and its Ricci tensor is bounded by below by $(n-1)g$, then the diameter of M is less or equal than π. In [BL96], the authors has proven that this kind of lower bound can be shown in a great generality, on a probability measure space with a Markov generator which satisfies a curvature-dimension condition. Moreover, the proof relies only on analytical tools, in particular on the existence of a Sobolev inequality of the form (2.12) and on the choice of the appropriate test functions (see Section 2 in [BL96] for the details).

The previous theorem gives us the Sobolev inequality needed to apply D. Bakry and M. Ledoux's proof. As a consequence, the Myers theorem holds on admissible stratified spaces in the following sense:

Theorem 2.14 (Singular Myers Theorem). *Let (X, g) an admissible stratified space. Let us define its Lipschitz diameter as:*

$$\mathrm{diam}_L(X) = \sup\left\{||\tilde{f}||_{L^\infty(X\times X}; f \in \mathrm{Lip}_1(X)\right\}$$

where $\tilde{f}(x,y) = f(x) - f(y)$ and $\mathrm{Lip}_1(X)$ is the set of Lipschitz functions with Lipschitz constant less or equal than one. Then $\mathrm{diam}_L(X)$ is less or equal than π.

Observe that on a smooth Riemannian manifold, what we called Lipschitz diameter coincides with the usual diameter associated to the Riemannian metric. We remark that it is possible to prove the following lemma:

Lemma 2.15. *Let (X, g) be a stratified space of dimension n with $Ric_g \geq (n-1)g$, and let $\gamma : [0,1] \to X$ be a Lipschitz curve in X. Let $L_g(\gamma)$ denote its length. For any $\varepsilon > 0$ there exists a curve γ_ε such that $\gamma_\varepsilon((0,1))$ is contained in the regular set Ω and $L_g(\gamma_\varepsilon) \leq (1+\varepsilon)L_g(\gamma)$.*

This implies two facts: first, a function u in $C^1(\Omega)$ whose gradient is bounded in $L^\infty(X)$ by a constant c is a Lipschitz function on the whole of X, with Lipschitz constant less or equal than c; moreover, the Lipschitz diameter coincides with the diameter associated to the metric g, and we can avoid any distinction between the two.

We are going to show that an admissible stratified space has diameter equal to π if and only if the first non-zero eigenvalue of the Laplacian is equal to the dimension of the space. Thanks to Theorem 4 in [BL96]this is in turn equivalent to the existence of extremal functions for the Sobolev inequality (2.12) which only depend on the distance from a point.

Theorem 2.16. *Let (X, g) be an admissible stratified space of dimension n. Then the following statements are equivalent:*

(i)) *The first non-zero eigenvalue of the Laplacian Δ_g is equal to n.*

(ii) *The diameter of X is equal to π.*

(iii) *There exist extremal functions for the Sobolev inequality.*

Proof. If the diameter of X is equal to π, then its Lipschitz diameter is equal to π, and then Theorem 4 in [BL96] implies both the existence of functions attaining the equality in Sobolev inequality and of an eigenfunction associated to the eigenvalue n. As a consequence, we have to prove that if the first non-zero eigenvalue of the Laplacian is equal to the dimension of the space, then its diameter is equal to π. If we find a Lipschitz function f which takes values in an interval of length π and whose Lipschitz constant is smaller or equal than one, then we have that $\mathrm{diam}_L(X) = \pi$, and thanks to the previous lemma we get the desired value for the diameter.

Consider φ an eigenfunction associated to the eigenvalue n: we have seen in the proof of Theorem 2.5 that its gradient belongs to $W^{1,2}(X)$ and that it is bounded. Moreover, its Hessian is proportional to the metric g: in fact, it must satisfy $|\nabla d\varphi|^2 = (\Delta_g \varphi)^2/n$, therefore we are in the case of equality in the Cauchy-Schwarz inequality and we get:

$$\nabla d\varphi = -\varphi g. \tag{2.22}$$

As a consequence, we can show that the quantity $|\nabla\varphi|^2 + \varphi^2$ is a constant on the regular set Ω. In fact we have:

$$d(|\nabla\varphi|^2 + \varphi^2) = 2\varphi d\varphi + 2\nabla d\varphi(\cdot, \nabla\varphi) = 2\varphi d\varphi - 2\varphi d\varphi = 0.$$

Then, up to multiplying by a constant, we can assume without loss of generality that:

$$|\nabla\varphi|^2 + \varphi^2 = 1 \quad \text{ on } \Omega. \tag{2.23}$$

This equality tells us that φ takes values between -1 and 1. Let us consider the function f defined as follows:

$$f = \arcsin(\varphi).$$

Its gradient is bounded on X, because the gradient of φ belongs to $L^\infty(X)$, and then f belongs to $\mathrm{Lip}(X)$ as well. Moreover, by definition ∇f has norm equal to one at each regular point: thanks to Lemma 2.15 this implies that the Lipschitz constant of f on the whole X is less or equal than one. In order to conclude, we need to show that the image of X by f is equal to $[-\pi/2, \pi/2]$. This is clearly equivalent to proving that φ has the closed interval $[-1, 1]$ as image.

Let us define $\mathcal{U}_+$ as the set on which φ is strictly positive. Observe that $\Omega \cap \mathcal{U}_+$ is not empty, since φ changes sign on X, and Ω is dense in X. Moreover $\Omega \cap \mathcal{U}_+$ is dense in $\mathcal{U}_+$, since Ω is dense and $\mathcal{U}_+$ is an open set in X.

Consider and the following problem with Dirichlet condition at the boundary:

$$\begin{cases} \Delta_g f = \lambda f \text{ in } \mathcal{U}_+ \\ f = 0 \text{ on } \partial\mathcal{U}_+. \end{cases}$$

This problem has a variational formulation: we can define the first non-zero Dirichlet eigenvalue on $\mathcal{U}_+$ as the infimum of the Dirichlet energy on functions in $W_0^{1,2}(\mathcal{U}_+)$, that is:

$$\lambda_1(\mathcal{U}_+) = \inf\left\{ \mathcal{E}(\psi) = \frac{\|d\psi\|_2^2}{\|\psi\|_2^2}, \psi \in W_0^{1,2}(\mathcal{U}_+) \right\}$$

Assume by contradiction that the maximum of φ is equal to M, strictly smaller than 1. We state that this implies the existence of a function $u : [0, M] \to \mathbb{R}_+$ such that $u(0) = 0$ and

$$\Delta_g(u \circ \varphi) = n\varphi u'(\varphi) - (1-\varphi^2)u''(\varphi) > n(u \circ \varphi), \text{ on } \Omega \cap \mathcal{U}_+.$$

This means that we can find a function u which vanishes at 0, is positive on $(0, M]$ and satisfies the following differential inequality on $(0, M]$:

$$-u''(t)(1-t^2) + ntu'(t) > nu(t). \tag{2.24}$$

Let $\alpha > 1$, to be chosen later, and consider $u_\alpha(t) = t - t^\alpha$. By replacing in the differential inequality, we reformulate (2.24) in the following way:

$$\alpha(\alpha-1)t^{\alpha-2}(1-t^2) + nt(1-\alpha t^{\alpha-1}) > n(t - t^\alpha).$$
$$\alpha(\alpha-1)t^{\alpha-2} - \alpha(\alpha-1)t^\alpha - n\alpha t^\alpha + nt^\alpha > 0$$
$$\alpha(\alpha-1)t^{\alpha-2} - (\alpha-1)t^\alpha(\alpha+n) > 0.$$

Now by multiplying by $(\alpha-1)t^{2-\alpha}>0$ we get:

$$\alpha - t^2(\alpha+n) > 0.$$

Therefore the question becomes to find an $\alpha>1$ such that the previous inequality is satisfied. The second degree polynomial appearing in the left-hand side of the previous inequality has a solution in $[0,1]$ at $t_0(\alpha)=\sqrt{\alpha(\alpha+n)^{-1}}$, and it is positive between 0 and $t_0(\alpha)$. Since this last quantity tends to one as α goes to infinity, and since M is strictly smaller than one, we can choose α large enough so that $t_0(\alpha)$ is strictly larger than M. For such α the function u_α satisfies the desired differential inequality, it is positive in $(0,M]$ and vanishes at 0. From now on we denote u_α simply by u, and $u\circ\varphi$ by ϕ.

Let ε be a positive real number and define $u_\varepsilon = u+\varepsilon$: then u_ε is strictly positive and, if we consider $\phi_\varepsilon = u_\varepsilon\circ\varphi$, the Laplacian of ϕ_ε satisfies $\Delta_g\phi_\varepsilon > n\phi$ on $\Omega\cap\mathcal{U}_+$.

For any positive function ψ belonging to $W_0^{1,2}(\mathcal{U}_+)$ we can define $v=\psi/\phi_\varepsilon$, which still belongs to $W_0^{1,2}(\mathcal{U}_+)$. By integration by parts and using that $\Omega\cap\mathcal{U}_+$ is dense in $\mathcal{U}_+$ we obtain:

$$\begin{aligned}
\int_{\mathcal{U}_+}|d\psi|^2dv_g &= \int_{\mathcal{U}_+}|d(v\phi_\varepsilon)|^2dv_g = \int_{\mathcal{U}_+}(v^2|d\phi_\varepsilon|^2+2v\phi_\varepsilon(dv,d\phi_\varepsilon)_g+\phi_\varepsilon^2|dv|^2)dv_g\\
&\geq \int_{\mathcal{U}_+}\phi_\varepsilon v^2\Delta_g\phi_\varepsilon dv_g = \int_{\mathcal{U}_+\cap\Omega}\phi_\varepsilon v^2\Delta_g\phi_\varepsilon dv_g > n\int_{\mathcal{U}_+\cap\Omega}\phi_\varepsilon\phi v^2 dv_g\\
&= n\int_{\mathcal{U}_+\cap\Omega}\psi^2\frac{\phi}{\phi_\varepsilon}dv_g = n\int_{\mathcal{U}_+}\psi^2\frac{\phi}{\phi_\varepsilon}dv_g.
\end{aligned}$$

Now observe that ϕ/ϕ_ε is smaller than one, it converges to one almost everywhere when ε goes to zero, and when we pass to the limit, by the dominated convergence theorem we, get:

$$\int_{\mathcal{U}_+}|d\psi|^2dv_g \geq n\int_{\mathcal{U}_+}\psi^2dv_g.$$

This shows that $\lambda_1(\mathcal{U}_+)$ is larger than or equal to n.

The eigenfunction φ associated to n is a positive function on $\mathcal{U}_+$ belonging to $W^{1,2}(\mathcal{U}_+)$, and therefore $\lambda_1(\mathcal{U}_+)$ is equal to n. Moreover, we can apply the same calculations as above with $\psi=\varphi$. We can write φ as $v\phi$, where v is strictly positive on $\mathcal{U}_+$ and it is defined by $v=(1-\varphi^{\alpha-1})^{-1}$, since by definition $\phi=\varphi-\varphi^\alpha$. We can easily deduce that v must be a positive constant. In fact we have:

$$n\int_{\mathcal{U}_+}\varphi^2dv_g = \int_{\mathcal{U}_+}|d\varphi|^2dv_g = \int_{\mathcal{U}_+}(\phi^2|dv|^2+\phi v^2\Delta_g\phi)dv_g > \int_{\mathcal{U}_+}\phi^2|dv|^2dv_g + n\int_{\mathcal{U}_+}\varphi^2dv_g.$$

This means that $dv=0$, v must be equal to a constant c and ϕ is a multiple of φ, therefore an eigenfunction relative to n. This is a contradiction, since we have shown that $\Delta_g\phi$ is strictly larger than $n\phi$ on $\Omega\cap\mathcal{U}_+$. Therefore, the maximum of ϕ on $\mathcal{U}_+$ must be equal to one.

Analogously, the minimum of φ is equal to -1: therefore the image of X via φ is $[-1, 1]$,, and via f is $[-\pi/2, \pi/2]$. Thanks to Theorem 2.14 we know that the Lipschitz diameter is less or equal than π, therefore we get the equality, as we wished. $\square$

Chapter 3

The Yamabe problem.

In this chapter we briefly present some known result about a foundational problem in geometric analysis: the Yamabe problem. The question posed by H. Yamabe in the sixties is whether on a compact smooth Riemannian manifold (M^n, g) of dimension n larger than 3 there exists a metric with constant scalar curvature within the conformal class of g. The motivation for considering n larger than three came from the fact that Poincaré's uniformization theorem holds in dimension two, and on any compact Riemannian surface there exists a metric with constant Gauss curvature. The answer to H. Yamabe's question is positive, and it is due to the works of several mathematicians, T. Aubin, N. Trudinger, R. Schoen, along twenty years. In our presentation we emphasise the study of a conformal invariant, the Yamabe constant, and of its relationship with the Yamabe constant of the sphere.

In the second section of this chapter we summarize the main results about the Yamabe problem on stratified space, which are contained in the recent paper by K. Akutagawa, G. Carron and R. Mazzeo [ACM14], and which have been the starting point of this thesis. In this case too, we are mainly interested in the role played by a conformal invariant, the *local* Yamabe constant, on which the existence of a metric with constant scalar curvature strongly depends.

3.1 The classical Yamabe problem

The study of the Yamabe problem on a compact smooth manifold has been the object of many mathematics works, and a complete presentation of its resolution is given in different sources: for example, we refer to the book [SY94] of R. Schoen and S.T. Yau and to the exhaustive survey of J.M. Lee and T.H. Parker [LP87]. We collect here the definitions and the results that we will need in the next section and chapter, in order to fix the notation and to underline the common points between the classical variational approach and the recent developments about the Yamabe problem on stratified spaces.

Consider a compact smooth manifold M of dimension $n \geq 3$, endowed with a Riemannian metric g. We define the *conformal class* of g the set of all metrics $\tilde{g}$ such that there exists a smooth function $f \in C^\infty(M)$ for which $\tilde{g} = e^{2f} g$. We denote the conformal class by $[g]$, so that H. Yamabe question can be reformulated: does there exist a metric $\tilde{g}$ in the conformal class $[g]$ which has constant scalar curvature $S_{\tilde{g}}$?

We recall the formula for the scalar curvature under conformal change, which can be found in Chapter 1-J of [Bes08]:

$$S_{\tilde{g}} = e^{-2f}(S_g + 2(n-1)\Delta_g f - (n-2)(n-1)|df|^2).$$

If we write a conformal metric in the form $\tilde{g} = u^{\frac{4}{n-2}} g$, for a smooth positive function u, the previous formula gets simpler and it reads:

$$S_{\tilde{g}} = u^{1-p}\left(\Delta_g u + a_n S_g u\right)$$

where we have

$$p = \frac{2n}{n-2}, \quad a_n = \frac{(n-2)}{4(n-1)}.$$

From now on in this chapter p will denote the exponent $2n/(n-2)$. The operator $L_g = \Delta_g + a_n S_g$ is referred to as the *conformal Laplacian*. As a consequence of this formulation, if it is possible to find a positive smooth function u solving the Yamabe equation:

$$L_g u = \Delta_g u + a_n S_g u = \lambda u^{\frac{n+2}{n-2}} \tag{3.1}$$

for a constant λ, then the metric $\tilde{g} = u^{\frac{4}{n-2}} g$ has constant scalar curvature.

A classical way to try and solve this equation, is to reformulate the problem in a variational way, and to see the Yamabe equation as the Euler-Lagrange equation associated to the appropriate functional. In fact, we can define the Yamabe functional Q_g. For a function u in $C^\infty(M)$ we consider

$$Q_g(u) = \frac{\displaystyle\int_M (|du|^2 + a_n S_g u^2) dv_g}{\|u\|_p^2} = \frac{E_g(u)}{\|u\|_p^2}. \tag{3.2}$$

Note that, again by using the conformal transformation of the scalar curvature, the Yamabe functional can be rewritten as a functional defined on the conformal class of g. For any $\tilde{g}$ in $[g]$ we have:

$$Q_g(\tilde{g}) = \frac{\displaystyle\int_M S_{\tilde{g}} dv_{\tilde{g}}}{\mathrm{Vol}_{\tilde{g}}(M)^{\frac{2}{n}}}. \tag{3.3}$$

A critical point of the Yamabe functional is a solution to the Yamabe equation. We can then define the Yamabe constant $Y(M^n, [g])$ as the infimum of Q_g over the conformal class:

$$Y(M, [g]) = \inf_{\tilde{g} \in [g]} Q_g(\tilde{g}) = \inf_{\substack{u \in C^\infty(M) \\ u \neq 0}} \frac{\displaystyle\int_M (|du|^2 + a_n S_g u^2) dv_g}{\|u\|_p^2}. \tag{3.4}$$

This is clearly a conformal invariant, and if there exists a smooth positive function u (equivalently, a metric $\tilde{g}$ in the conformal class) attaining the Yamabe constant, then u is a solution to the Yamabe equation. In this case the metric

$$\tilde{g} = u^{\frac{4}{n-2}} g.$$

has constant scalar curvature, and we refer to it as a Yamabe metric. A metric with constant scalar curvature is not necessarily a Yamabe metric, since it is not necessarily minimizing. Note also that $C^\infty(M)$ is dense in the Sobolev space $W^{1,2}(M)$ and therefore in the definition of the Yamabe functional and of the Yamabe constant we can consider functions u belonging to $W^{1,2}(X)$ instead of only smooth functions.

Observe that the difficulty of the Yamabe equation relies on the fact that the exponent $p-1$ in the right-hand side of (3.1) leads to the critical exponent for which the compactness of the Sobolev embedding $W^{1,2}(M) \hookrightarrow L^p(M)$ fails. The works of N. Trudinger and T. Aubin showed that this issue can be passed over if we have further information on the Yamabe constant $Y(M,[g])$. The first one proved that if the Yamabe constant is non positive, then there exists a unique solution to the Yamabe equation and therefore a conformal metric with constant scalar curvature.

T. Aubin proved in [Aub76a] a fundamental inequality which we refer to as Aubin's inequality:

Theorem 3.1 (Aubin's inequality). *The Yamabe constant of any compact manifold $(M^n,[g])$ is less than or equal to the Yamabe constant Y_n of the sphere $\mathbb{S}^n$ endowed with the standard metric:*

$$Y(M^n,[g]) \leq Y(\mathbb{S}^n,[can]) = Y_n.$$

Note that the proof of this result depends on a local argument and on the fact that the Yamabe constant of the sphere Y_n coincides with the best constant Λ in the following Sobolev inequality in $\mathbb{R}^n$:

$$\Lambda \left(\int_{\mathbb{R}^n} u^p dx \right)^{\frac{2}{p}} \leq \int_{\mathbb{R}^n} |du|^2 dx.$$

As a consequence, Aubin's inequality holds also for complete smooth manifolds.

What is crucial in solving the Yamabe problem with the variational approach is the following result:

Theorem 3.2 (T. Aubin). *Let (M^n,g) be a compact Riemannian manifold of dimension $n \geq 3$. If Aubin's inequality is strict, that is $Y(M^n,[g]) < Y_n$, then there exists a solution to the Yamabe equation.*

We briefly recall the steps that prove the previous theorem. The main idea consists in considering a sub-critical equation, in which we replace the critical exponent by a smaller one, and to show that the sub-critical solutions converge uniformly to a smooth positive function solving the Yamabe equation. The crucial point is to show that the sequence of sub-critical solutions is uniformly bounded in $L^\infty(M)$, this allowing to deduce the uniform convergence.

Then, for $s \in [2, p)$ we define the sub-critical functional:

$$Q_s(u) = \frac{E(u)}{\|u\|_s^2} \qquad \lambda_s = \inf\{Q_s(u), u \in W^{1,2}(M), u > 0\} \tag{3.5}$$

Standard arguments in calculus of variations imply that there exists a positive function u_s belonging to $W^{1,2}(M)$, with $\|u_s\|_s = 1$, which attains the infimum λ_s of the sub-critical functional: therefore, u_s is a weak solution of the following equation:

$$\Delta_g u_s + a_n S_g u_s = \lambda_s u_s^{s-1}$$

Elliptic regularity, the Morrey and Schauder inequalities allow one to deduce that u_s is in fact smooth on M, thus u_s is a strong solution of the previous equation. Moreover, it is not difficult to prove that the sequence $\{\lambda_s\}_s$ is such that:

$$\liminf_{s \to p} \lambda_s = Y(M, [g]).$$

When $Y(M, [g]) \geq 0$, $\{\lambda_s\}_s$ converges exactly to the Yamabe constant of the manifold. If the sequence $\{u_s\}_s$ converges uniformly to some smooth positive function u on M, then u is such that

$$Q_g(u) = \lim_s \lambda_s \leq Y(M, [g])$$

By definition of the Yamabe constant, we then obtain $Q_g(u) = Y(M, [g])$. It remains then to prove the uniform convergence of the sequence of sub-critical solutions.

If by contradiction the sequence $\{u_s\}_s$ is not bounded, we can find, up to a subsequence, a sequence of points x_s in M such that $u_s(x_s)$ tends to infinity as s tends to p. Since M is compact, there exists a point x_0 in M to which the x_s converges as s tends to p. We consider normal coordinates in a small neighbourhood of x_0 and, by looking the sub-critical equation for u_s in these coordinates, we are able to find a smooth positive function v on $\mathbb{R}^n$ such that $\|v\|_2 \leq 1$, solving the following equation:

$$\Delta_0 v = \lambda v^{p-1}, \quad \lambda = \lim_s \lambda_s.$$

where Δ_0 is the Euclidean Laplacian in $\mathbb{R}^n$. By using cut-off function to approach v by compact supported functions, it is not difficult to show that the previous equation implies that $\lambda \geq 0$, then $\lambda = Y(M, [g])$ and moreover

$$Y(M, [g]) = \lambda \geq \Lambda = Y_n.$$

This contradicts our hypothesis. As a consequence, the sequence $\{u_s\}_s$ must be uniformly bounded, and then by a regularity argument it converges uniformly in $C^k(M)$ for any positive k to a smooth solution u of the Yamabe equation. Thanks to the maximum principle, this latter cannot be the null solution and it is positive. This concludes the proof of the existence when $Y(M, [g]) < Y_n$.

T. Aubin's contribution to the solution of the Yamabe problem also includes the proof that when the dimension of the manifold is greater than 6, and the metric is not locally conformally flat, the strict inequality, and therefore the existence of a solution to the Yamabe equation, hold.

For what concerns the other cases, the local argument proposed by T. Aubin does not suffice to conclude. Around a decade later, the works of S.T. Yau and R. Schoen (see [SY79], [SY81], [SY88]) about the positive mass conjecture in dimension $n \leq 6$ or locally conformally flat metric, allowed this latter to deduce a rigidity result [Sch84]: if the Yamabe constant of (M^n, g) coincides with the one of the sphere Y_n, then the manifold is conformally equivalent to the sphere. As a consequence, it naturally carries a metric with constant scalar curvature. N. Trudinger, T. Aubin and R. Schoen's results taken together give a complete answer to the Yamabe problem on compact manifolds: we summarize them in the following theorem.

Theorem 3.3 (N. Trudinger, T. Aubin, R. Schoen). *Let (M^n, g) a compact Riemannian manifold of dimension $n \geq 3$. Either the Yamabe constant of (M^n, g) is strictly smaller than Y_n, or (M^n, g) is conformally equivalent to the sphere. In both cases, there exists a metric of constant scalar curvature in the conformal class of g.*

Other approaches exist in order to solve the Yamabe problem, for example by studying the Yamabe flow, which exists for any time, and its long time convergence. We refer to the complete survey by S. Brendle [Bre08] for a description of what is already known in this direction.

The advantage of the variational approach is that it is easily adaptable to many different contexts. In the following we focus on the Yamabe problem on stratified spaces, but we underline that there are many variants of the Yamabe problem. The same question of finding a conformal metric with constant scalar curvature can be considered on open manifolds, on subdomains of the sphere (the singular Yamabe problem), on manifolds with boundary, within conformal metrics that are invariant under the action of a subgroup of isometries (the G-invariant Yamabe problem), on complex manifolds (the Chern-Yamabe problem)... In the next chapter we will consider for example the Yamabe problem on a class of complete open manifolds, almost homogeneous manifolds. We are far from giving an exhaustive list of the possible existing developments of H. Yamabe's original question, and there are surely more variations to come in the future.

Moreover, a related and extremely interesting question is the study of the Yamabe invariant of a manifold M^n. If we denote by $\mathcal{M}$ the set of all Riemannian metrics on a differentiable manifold M^n, we define the Yamabe invariant:

$$Y(M) = \sup_{g \in \mathcal{M}} Y(M, [g]) = \sup_{g \in \mathcal{M}} \inf_{\tilde{g} \in [g]} Q_g(\tilde{g}).$$

A Riemannian metric attaining the Yamabe invariant, if it exists, is an Einstein metric. The existence of an Einstein metric is a much more difficult question than the Yamabe problem, but this latter can be considered as a first step through a min-max procedure to construct Einstein metrics on compact manifolds.

3.2 The Yamabe problem on stratified spaces

The following discussion is based on the article by K. Akutagawa, G. Carron and R. Mazzeo [ACM14]. The authors formulate the Yamabe problem in the very general context of *almost smooth metric-measure spaces*, to which all the first section of the article is devoted. In order not to add more definitions that we will not use, we only consider the case of stratified spaces.

Assumptions and Yamabe constant

Given an iterated edge metric g on a stratified space (X, g) of dimension $n \geq 3$, we can consider the Yamabe functional and try to find its critical points among the functions belonging to the Sobolev space $W^{1,2}(X)$. A first question is whether $Q_g(u)$ is well-defined for any function u in $W^{1,2}(X)$.

The Sobolev embeddings we recalled in Chapter 1 ensure that a function in $W^{1,2}(X)$ has finite norm in $L^p(X)$, and therefore the denominator of Q_g does not give any problem. As for the numerator $E_g(u)$, it is necessary to assume an integrability condition on the scalar curvature of g. Accordingly to [ACM14], we assume that one of the following assumption holds:

(a) S_g belongs to $L^q(X)$ for $q > \frac{n}{2}$;

(b) For some $q > 1$ there exist $\alpha \in [0, 2)$ and a positive constant C such that for any point x of X we have:

$$\sup_{r>0} \left(r^{\alpha q - n} \int_{B(x,r)} |S_g|^q dv_g \right) \leq C.$$

Observe that the condition (a) implies (b) with α equal to $n/q < 2$ and C equal to the norm in $L^q(X)$ of V. In both of these cases the term containing the scalar curvature in $E_g(u)$:

$$\int_X S_g u^2 dv_g.$$

is finite for any u in $W^{1,2}(X)$. When S_g satisfies the hypothesis (a) this is simply a consequence of Hölder inequality; otherwise we refer to Lemma 1.1 in [ACM14] to state that for any $\varepsilon > 0$ there exists a positive constant C_ε such that for all $u \in W^{1,2}(X)$ we have:

$$\int_X |S_g||u|^2 dv_g \leq \varepsilon \int_X |du|^2 dv_g + C_\varepsilon \int_X |u|^2 dv_g < +\infty. \tag{3.6}$$

As a consequence, under one of the assumptions (a) or (b) the Yamabe functional Q_g is well defined, we can study its critical points and consider the Yamabe constant of a stratified space:

$$Y(X, [g]) = \inf_{u \in W^{1,2}(X)} Q_g(u).$$

In the classical case, the existence of a solution to the Yamabe equation depends on the relationship between the Yamabe constant of the manifold and the Yamabe constant of the sphere of corresponding dimension. There is an analogous situation in the case of stratified spaces: we have to compare the Yamabe constant of X with a conformal invariant and get an equivalent of Aubin's inequality. Such conformal invariant is no longer Y_n, but the *local Yamabe constant.*

The local Yamabe constant and the existence result

We first define the Yamabe constant of an open set $\mathcal{U} \subset X$:

$$Y(\mathcal{U}) = \inf\{Q_g(u), \text{ for } u \in W_0^{1,2}(\mathcal{U} \cap \Omega)\}$$

where Ω is the regular set of X. Observe that for any open set $\mathcal{U}$ in X we clearly have $Y(X,[g]) \leq Y(\mathcal{U})$. Then we give the following definition:

Definition 9 (Local Yamabe Constant)**.** The local Yamabe constant of a stratified space (X, g) is defined as:

$$Y_\ell(X) = \inf_{x \in X} \lim_{r \to 0} Y(B(p,r)).$$

The generalized Aubin's inequality $Y(X,[g]) \leq Y_\ell(X)$ holds for any stratified space.

Note that if we consider the previous definition on a compact smooth manifold (M^n, g), the local Yamabe constant of (M^n, g) coincides with the one of the sphere Y_n, so that the local Yamabe constant appears to be a good generalization of that conformal invariant in the case of singular spaces. In particular, we have that the local Yamabe constant of a stratified space is less than or equal to Y_n. A preceding version of the local Yamabe constant can be found in the cylindrical Yamabe constant defined by K. Akutagawa and B. Botvinnik in [AB03], which can be considered as the local Yamabe constant associated to a stratified space with only isolated conical singularities and depth equal to one.

By using the local description of the iterated edge metric, if X has strata X^j with links (Z_j, k_j) it is possible to show that the local Yamabe constant is equal to:

$$Y_\ell(X) = \min_j \inf_{x \in X^j} \{Y(\mathbb{R}^j \times C(Z_j), [dy^2 + dr^2 + r^2 k_{j,x}]\}. \tag{3.7}$$

We know from Lemma 1.4 that $\mathbb{R}^j \times C(Z_j)$ is isometric to the cone on the tangent sphere, so that the local Yamabe constant can be also be computed as:

$$Y_\ell(X) = \inf_{x \in X} Y(C(S_x), [dt^2 + t^2 h_x]).$$

Moreover, if we factorise r^2 in the product metric $dy^2 + dr^2 + r^2 k_{j,x}$ it is not difficult to see that $\mathbb{R}^j \times C(Z_j)$ is conformal to the product of an hyperbolic space and Z_j. Then we can also express the local Yamabe constant in the following form:

$$Y_\ell(X) = \min_j \inf_{x \in X^j} \{Y(\mathbb{H}^{j+1} \times Z_j, [g_{\mathbb{H}^{j+1}} + k_{j,x}]\}.$$

In [ACM14], it is shown (see Proposition 1.4(b)) that under the assumption (a) or (b) on the scalar curvature the local Yamabe constant is positive. Moreover, for any $\varepsilon > 0$ there exists a constant C_ε such that the following Sobolev inequality holds for any u is $W^{1,2}(X)$:

$$(Y_\ell(X) - \varepsilon) \|u\|^2_{\frac{2n}{n-2}} \leq \int_\Omega |du|^2 dv_g + C_\varepsilon \int_\Omega u^2 dv_g. \tag{3.8}$$

The authors also prove the following existence theorem:

Theorem 3.4. *Let (X, g) be a stratified space of dimension n. Assume the following:*

(i) *The scalar curvature S_g satisfies one of the assumptions* (a) *or* (b)*;*

(ii) *The strict generalized Aubin's inequality holds:*

$$Y(X, [g]) < Y_\ell(X).$$

Then there exists a bounded, non-negative function u in $W^{1,2}(X)$ which attains the Yamabe constant $Y(X, [g])$. Moreover, on the regular set Ω, the function u solves the Yamabe equation.

We briefly sketch the proof. In the next chapter the proof of Theorem 4.20 follows the same technique as the following: we will give more details in that context.

First, it is possible to prove the result for the regularity of subsolutions to a Schrödinger equation:

Proposition 3.5. *Assume that V is a function on X satisfying one of the hypothesis* (a) *or* (b)*. Let u in $W^{1,2}(X)$ be a non-negative function for which the weak inequality $\Delta_g u \leq Vu$ holds. Then u is bounded on X.*

If (a) holds, this in nothing but the Moser iteration technique that we recalled in Chapter 1. Otherwise, we refer to Theorem 1.10 in [ACM14].

Then one can consider the sub-critical Yamabe quotient Q_s for $s \in [2, p)$ as in the classical case, and find a sequence of sub-critical solutions $\{u_s\}_s$ which are non-negative and, thanks to Proposition 3.5 with the assumption (b), bounded. Moreover $Y_s = Q_s(u_s)$ converges to the Yamabe constant $Y(X, [g])$ as s tends to p, and thus the norm in $W^{1,2}(X)$ of u_s is uniformly bounded. Therefore, thanks to the Sobolev embeddings, u_s converges to a function $u \in W^{1,2}(X)$ weakly in $W^{1,2}(X)$ ans strongly in any $L^q(X)$ for $q \in [1, p)$.

Consider the weak form of the equation satisfied by u_s: for any test function φ in $C^1_0(\Omega)$ we have

$$\int_X ((du_s, d\varphi) + S_g u_s \varphi) dv_g = \int_X u_s^{s-1} \varphi dv_g.$$

The weak convergence in $W^{1,2}(X)$ assures that we can pas to the limit as s tends to p in the first term of the left-hand side. If the convergence of u_s is strong in $L^p(X)$ as well, we can pass to the limit in the other terms too, and deduce that u is a weak solution for the Yamabe equation. Classical regularity arguments show that u is also a strong solution on the regular set Ω of X. Therefore the rest of the proof consists in proving that $\{u_s\}_s$ is a bounded sequence in $L^\infty(X)$: since X is compact, the dominated convergence theorem will immediately imply the strong convergence in $L^p(X)$. This is done by applying again Proposition 3.5 to the sub-critical equation:

$$\Delta_g u_s = (Y_s u_s^{s-2} - a_n S_g)u = V_s u_s, \quad V_s = Y_s u_s^{s-2} - a_n S_g.$$

In order to do that, the potential V_s must satisfy one of the assumption (a) or (b). If we apply the first step of Moser iteration technique to u_s, the Sobolev inequality (3.8) and the fact that Y_s remains strictly smaller than $Y_\ell(X)$, then we can deduce that u_s belongs to $L^{\alpha p}(X)$ for some α strictly larger than one. As a consequence u_s to the power $(s-2)$ belongs to $L^{\alpha n/2}(X)$, and in particular satisfies both the assumptions (a) and (b). Therefore we can apply Proposition 3.5 with the potential V_s and get a uniform bound on the norm in $L^\infty(X)$ of u_s:

$$\|u_s\|_\infty \leq C' \|u_s\|_{1,2} \leq C. \tag{3.9}$$

As we said before this implies the convergence in $L^p(X)$, and therefore we can pass to the limit to deduce that the function u in $W^{1,2}(X) \cap L^\infty(X)$ attains the Yamabe constant of X and it is a solution to the Yamabe equation on Ω. This concludes the proof.

Remark 3.6. In Theorem 1.12 in [ACM14], one more case is studied. Instead of assuming (a) or (b), it is possible to consider only the negative part S_g^- of the scalar curvature and ask that it belongs to $L^q(X)$ for $q > n/2$. In this case $Y_\ell(X)$ is not necessarily positive, and in the generalized Aubin inequality it is replaced by the *local Sobolev constant* $S_\ell(X)$. Nothing changes in the proof of the existence theorem apart from considering S_g^- instead of S_g in V_s.

If we want to apply the previous existence theorem, we first need the hypothesis on the scalar curvature to be satisfied. This issue is extensively studied in Section 2.3 of [ACM14]. Let (X, g) be a stratified space with strata X^j, each with link Z_j of dimension $d_j = (n-j-1)$. If the iterated edge metric g is assumed to be smooth in the variable that we denoted ρ_j near each stratum, then its scalar curvature in a neighbourhood of X^j has the form:

$$S_g = \frac{A_0^j(\rho_j)}{\rho_j^2} + \frac{A_1^j(\rho_j)}{\rho_j} + \mathcal{O}(1).$$

By combining this expression with the one of the volume element near X^j, it is possible to show that:

(1) the scalar curvature S_g satisfies (b) if and only if A_0^j vanishes for all j;

(2) the scalar curvature S_g satisfies (a) if and only if A_0^j vanishes for all j and A_1^j vanishes for all j such that the dimension of the link d_j is less than or equal to $(n-2)/2$.

It is then possible to refine the expression of the scalar curvature by using the formulas for the curvature of warped product metrics. This leads to the following result:

Theorem 3.7. *Let (X, g) be a stratified space with strata X^j and links (Z_j, k_j) of dimension d_j. If for all j the scalar curvature of the link S_{k_j} is equal to $d_j(d_j - 1)$, the scalar curvature S_g satisfies the assumption* (b).

The assumption on the scalar curvature of the links is very rigid, and in [ACM14] it is shown that another strategy to make the terms A_0^j vanish is to look for a metric $\tilde{g}$ in the conformal class of g for which A_0^j disappears. This is related to the positivity and discreteness of the spectrum of the following operator on the links:

$$\mathcal{L}_{k_j}^n = \Delta_{k_j} + a_n S_{k_j}.$$

Such operator differs from the conformal Laplacian on Z_j for the constant a_n instead of the one depending on the dimension of Z_j. It had been introduced in [AB03] in the context of open manifolds with cylindrical ends (that are conformal to isolated conical singularities). The role played by $\mathcal{L}_{k_j}^n$ is very interesting, but we will not use it in the following: we refer to [ACM14] and [AB03] for a detailed discussion.

For what concerns the second hypothesis, the explicit value of the local Yamabe constant is unknown for general stratified spaces, even for general isolated conical singularity, and little can be done without knowing it. The goal of the next chapter is to show how to compute the local Yamabe constant under a geometric assumption on the links.

Some results exist in the case of orbifolds with isolated conical singularities (see [AB04], [Aku12]). In [AB03] the authors consider what they call *canonical cylindrical manifolds*, that are products between a compact Riemannian manifold (Z^n, h) and $\mathbb{R}$, endowed with the product metric $dt^2 + h$, and are clearly conformal to a spherical suspension over Z^n. In this case the local Yamabe constant of the spherical suspension coincides with the cylindrical Yamabe constant of $Z \times \mathbb{R}$. It is shown in [AB03] that when the dimension of Z is larger than or equal to 5, and the metric is not conformally flat, the cylindrical Yamabe constant is always strictly less than the one of the sphere, and a Yamabe metric exists. In the other cases, for $n = 2, 3, 4$ and locally conformally flat metric, if the first non-zero eigenvalue of the operator $\mathcal{L}_h^{n+1}$ is positive, then there is a rigidity result: the cylindrical Yamabe constant of $Z \times \mathbb{R}$ is equal to the one of the sphere if and only if Z^n is homothetic to $\mathbb{S}^n$. Therefore, consider canonical cylindrical manifold $Z^n \times \mathbb{R}$, $n \geq 2$, or equivalently a spherical suspension $C(Z)$, such that

(i) Z^n is *not* homothetic to $\mathbb{S}^n$;

(ii) the first non-zero eigenvalue of the operator $\mathcal{L}_h^{n+1}$ is positive

Then the cylindrical Yamabe constant is strictly less than Y_n and a Yamabe metric exists. This gives at least some examples of stratified spaces on which the Yamabe problem is solvable.

Nevertheless, J. Viaclovsky in [Via10] has given an example of orbifold for which the Yamabe constant coincides with the local Yamabe constant and a Yamabe metric does *not* exists. He considers a manifold (X, g_{GB}) of dimension four endowed with a asymptotically locally euclidean Gibbons-Hawking metric, and its conformal compactification $(\hat{X}, \hat{g})$. He shows that this latter is smooth with one orbifold isolated singularity (p, Γ), and it has positive orbifold Yamabe invariant (with our terms positive local Yamabe constant) equal to:

$$Y(\hat{X}, \hat{g}) = Y_\ell(\hat{X}) = \frac{Y_4}{\sqrt{|\Gamma|}}$$

Furthermore, $(\hat{X}, \hat{g})$ does not admit a Yamabe metric. It would be interesting to find other cases of non existence, to understand better when they occur, and whether there is any rigidity phenomena analogous to the one holding for compact smooth manifolds.

Chapter 4

The local Yamabe constant of Stratified Spaces.

This chapter is devoted to showing that we are able to compute the local Yamabe constant of a stratified space if its links are endowed with an Einstein metric. This hypothesis is motivated by Theorem 3.7: if the scalar curvature of the links Z_j of dimension d_j is equal to $d_j(d_j-1)$, then the scalar curvature of the whole stratified space satisfies the integrability condition of the existence theorem 3.4. This is necessary if we hope to find examples in which the solution to the Yamabe equation exists.

Our first result is a lower bound on the Yamabe constant of an admissible stratified space: this is a direct consequence of the Sobolev inequality that we proved in Chapter 2. Such lower bound generalizes a result obtained by J. Petean [Pet09] about the Yamabe constant of cones over smooth compact manifolds with a positive Ricci lower bound. Moreover, our result gives the exact value of the Yamabe constant in the case of an Einstein metric. This agrees with a computation of the orbifold Yamabe invariant given by K. Akutagawa and B. Botvinnik in [AB04], and also gives the value of the local Yamabe constant of *any* orbifold, not necessarily with isolated singularities.

The previous result does not apply to a stratified space with a stratum of codimension two and cone angle larger than 2π. In this last case, we show that the Yamabe constant of the product between the Euclidean space $\mathbb{R}^{n-2}$ and a cone $C(\mathbb{R}/\alpha\mathbb{Z})$ of angle greater than 2π coincides with the Yamabe constant of the sphere of dimension n. The proof relies on a smoothing technique and on the study of the isoperimetric profiles of $\mathbb{R}^{n-2}\times C(\mathbb{R}/\alpha\mathbb{Z})$.

These results considered together give a complete answer to the question of computing the local Yamabe constant of a stratified space with Einstein links.

In the third part of this Chapter we present another approach to the problem of studying the local Yamabe constant, which is inspired by the works of M. Obata [Oba72] and J. Viaclovsky [Via10]. We show a rigidity result following from the singular version of Obata's theorem contained in Chapter 2: if there exists a non-trivial Yamabe minimizer on an Einstein stratified space, then this latter must be isometric to the spherical suspension of another stratified space of lower dimension, and again, the Einstein metric attains the Yamabe constant. This gives another path to compute the local Yamabe constant, provided that we are able to find a non-trivial solution to the Yamabe equation. We give a family of examples in which we can prove the existence of a Yamabe minimizer by presenting a new proof of a result due to K. Akutagawa and N. Grosse [Gro13].

In the following we will alternatively use the various conformal equivalences that we introduced in the previous chapter: the local Yamabe constant can be computed by studying the product $\mathbb{R}^j \times C(Z)$, or $\mathbb{H}^{j+1} \times Z$ or the cone over a m-fold spherical suspension of the link, for the appropriate m.

4.1 A lower bound for the Yamabe constant

In Theorem 2.13 we have shown that a Sobolev inequality with explicit constants depending only on the dimension and on the volume holds on an admissible stratified space. As a consequence of that result, we can prove a lower bound for the Yamabe constant of admissible stratified spaces, which is attained in the case of Einstein metrics, as the following proposition shows:

Proposition 4.1. *Let (X, g) be an admissible stratified space. Then its Yamabe constant is bounded by below:*

$$Y(X,[g]) \geq \frac{n(n-2)}{4} V^{\frac{2}{n}} \tag{4.1}$$

In particular, if g is an Einstein metric, we have equality.

Proof. Recall that the Yamabe constant of X is defined by

$$Y(X,[g]) = \inf_{u \in W^{1,2}(X), u \neq 0} \frac{\displaystyle\int_X (|du|^2 + a_n S_g u^2) dv_g}{\|u\|^2_{\frac{2n}{n-2}}}.$$

where $a_n = \frac{n-2}{4(n-1)}$ and S_g is the scalar curvature. Since $Ric_g \geq k(n-1)g$, we have $S_g \geq kn(n-1)$, and as a consequence

$$a_n S_g \geq \frac{n(n-2)}{4}.$$

We denote this constant by γ^{-1}. Remark that if we take $p = \frac{2n}{n-2}$ in the previous theorem, γ is exactly the constant appearing in the right-hand side of the Sobolev inequality of Theorem 2.13. Then for any $u \in W^{1,2}(X)$ we have:

$$\frac{V^{\frac{2}{n}}}{\gamma} \|u\|^2_{\frac{2n}{n-2}} \leq \|du\|_2^2 + \frac{1}{\gamma}\|u\|_2^2 \leq \|du\|_2^2 + \int_X a_n S_g |du|^2 dv_g.$$

and this easily implies the desired bound by below on the Yamabe constant.

Recall that in order to define the Yamabe constant (3.4) we can also consider the infimum of the Yamabe functional Q_g, as in (3.3), over the conformal class. When we consider an Einstein metric g on an admissible stratified space, its Yamabe quotient attains exactly

$$Q(g) = \frac{n(n-2)}{4} \operatorname{Vol}_g(X)^{\frac{2}{n}}$$

since the scalar curvature of g is constant and equal to $n(n-1)$. Thanks to our lower bound and the fact that the Yamabe constant is an infimum, we get the case of equality in the Einstein case. □

This result gives us the Yamabe constant of an admissible stratified space with Einstein metric; it can also be used to compute the *local* Yamabe constant of an admissible stratified space with Einstein links, as we show in the following examples.

4.1.1 Examples

Consider an admissible stratified space (Z^d, k) of dimension d with Einstein metric k. We know from Chapter 1 that $\mathbb{R}^{n-d-1} \times C(Z)$ endowed with the model metric is isometric to $C(S)$ with the exact cone metric $dr^2 + r^2 h$, where:

$$\begin{aligned} S &= \left[0, \frac{\pi}{2}\right] \times \mathbb{S}^{n-d-3} \times Z \\ h &= d\varphi^2 + \cos^2(\varphi) g_{\mathbb{S}^{n-d-3}} + \sin^2(\varphi) k. \end{aligned}$$

The exact cone metric is in fact conformal to the metric $g_c = dt^2 + \cos^2(t) h$, for $t \in [-\pi/2, \pi/2]$, which is an Einstein metric. We have then the following:

Lemma 4.2. *Let (Z,k) an admissible Einstein stratified space of dimension d. The product $X = \mathbb{R}^{n-d-1} \times C(Z)$ endowed with the metric $g = dy^2 + dr^2 + r^2 k$ is conformally equivalent to an Einstein admissible stratified space $(C(S), g_c)$.*

Proposition 4.1 states then that Yamabe constant of (X,g) will be equal to:

$$Y(X,[g]) = Y(C(S),[g_c]) = \frac{n(n-2)}{4} \operatorname{Vol}_{g_c}(C(S))^{\frac{n}{2}}. \tag{4.2}$$

By computing the volume of $C(S)$ with respect to g_c we get the following explicit value:

Lemma 4.3. *Let (Z^d, k) be an admissible stratified space of dimension d with Einstein metric k. Then the Yamabe constant of $X = \mathbb{R}^{n-d-1} \times Z$ endowed with the metric g as above is equal to:*

$$Y(X,[g]) = \left(\frac{\operatorname{Vol}_k(Z)}{\operatorname{Vol}(\mathbb{S}^d)}\right)^{\frac{2}{n}} Y_n. \tag{4.3}$$

Proof. Since the Yamabe constant of the sphere $\mathbb{S}^n$ is:

$$Y_n = \frac{n(n-2)}{4} \operatorname{Vol}(\mathbb{S}^n)^{\frac{n}{2}}.$$

we can reformulate (4.2) and get:

$$Y(C(S),[g_c]) = Y_n \left(\frac{\mathrm{Vol}_{g_c}(C(S))}{\mathrm{Vol}(\mathbb{S}^n)} \right)^{\frac{n}{2}}$$

It remains to compute the volume of $C(S)$ with respect to g_c, which is clearly equal to:

$$\mathrm{Vol}_{g_c}(C(S)) = 2\,\mathrm{Vol}_h(S) \int_0^{\frac{\pi}{2}} \cos^{n-1}(t)dt.$$

Now recalling that h has the form $h = d\theta^2 + \cos^2(\theta) g_{\mathbb{S}^j} + \sin^2(\theta)k$ for $j = n-d-3$, the volume of S with respect to h is:

$$\mathrm{Vol}_h(S) = \mathrm{Vol}(\mathbb{S}^{n-d-3})\,\mathrm{Vol}_k(Z) \int_0^{\frac{\pi}{2}} \cos^{n-d-3}(\varphi)\sin^d(\varphi)d\varphi.$$

By using polar coordinates, the sphere $\mathbb{S}^n$ can be viewed as the cone over the $(n-d-3)$-fold spherical suspension of $\mathbb{S}^d$, so that we can express its volume in the following form:

$$\mathrm{Vol}(\mathbb{S}^n) = 2\,\mathrm{Vol}(\mathbb{S}^{n-d-3})\,\mathrm{Vol}(\mathbb{S}^d) \int_0^{\frac{\pi}{2}} \cos^{n-d-3}(\varphi)\sin^d(\varphi)d\varphi \int_0^{\frac{\pi}{2}} \cos^{n-1}(t)dt.$$

Finally by replacing the two expressions for the volumes of $C(S)$ and $\mathbb{S}^n$ we get the desired value of $Y(C(S),[g_c])$. □

Simple edge of codimension 2: In the simplest case of Z being a circle of radius $a < 1$, $C(\mathbb{R}/\alpha\mathbb{Z})$ is a cone of angle $\alpha = 2\pi a$, and a similar calculation leads to:

$$Y(\mathbb{R}^{n-2} \times C(\mathbb{R}/\alpha\mathbb{Z}),[g]) = a^{\frac{2}{n}} Y_n = \left(\frac{\alpha}{2\pi}\right)^{\frac{2}{n}} Y_n.$$

Observe that this procedure cannot be applied if Z is a circle with radius larger than one, since in our definition of admissible stratified spaces we excluded the existence of codimension 2 strata with cone angle larger than 2π. In the next section we are going to give another way to compute the Yamabe constant of this kind of stratified spaces.

Orbifolds: The previous lemma can be applied in particular to a cone over a quotient of the sphere $\mathbb{S}^{n-1}$ for a finite group Γ acting freely. If we denote by g the metric induced by the quotient, we obtain:

$$Y(C(\mathbb{S}^{n-1}/\Gamma),[g]) = \left(\frac{\mathrm{Vol}_g(\mathbb{S}^{n-1}/\Gamma)}{\mathrm{Vol}\,\mathbb{S}^n} \right)^{\frac{2}{n}} Y_n = \frac{Y_n}{|\Gamma|^{\frac{2}{n}}}.$$

As a consequence, our result applies to *any* orbifold, whose singularities are both isolated or not, since at a singular point of an orbifold the link is always a quotient of the sphere for a finite group. Moreover, this agrees with a result of K. Akutagawa and B. Botvinnik contained in [AB04]. By reformulating Theorem 3.1 in [Aku12] we can state that the local Yamabe constant of an orbifold with isolated singularities $\{(p_1, \Gamma_1), \ldots, (p_s, \Gamma_s)\}$ is equal to:

$$Y_\ell(X) = \min_{j=1\ldots s} \frac{Y_n}{|\Gamma_j|^{\frac{2}{n}}}.$$

which is the same as what we found above.

Cones over smooth manifolds: Lemma 4.3 extends a result by J. Petean about the Yamabe constant of cones. The author shows in [Pet09] that if M is a compact manifold of dimension n, endowed with a Riemannian metric such that $Ric_g = (n-1)g$, then the Yamabe constant of the cone $C(M) = (0, \pi) \times M$ endowed with the cone metric $dt^2 + \sin^2(t)g$ is equal to:

$$Y(C(M), [dt^2 + \sin^2(t)g]) = \left(\frac{\mathrm{Vol}_g(M)}{\mathrm{Vol}(\mathbb{S}^n)}\right)^{\frac{2}{n+1}} Y_{n+1}.$$

If the spherical suspension S were a compact smooth manifold, our computation would give exactly the same result. Note that the argument of J. Petean is based on the study of isoperimetric domains in cones over manifolds with a positive Ricci lower bound. It would be interesting to generalize this technique to cones over admissible stratified spaces Ricci tensor bounded by below, when the dimension of the singular set is small enough (that is, the Hausdorff dimension of the singular set must be smaller than $n-3$).

4.2 Strata of codimension 2 and cone angle $\alpha > 2\pi$.

In order to have a complete answer to the question of computing the local Yamabe constant of a stratified space with Einstein links, it remains to study the case of codimension two strata with cone angle greater than 2π. The link of such strata is a circle of diameter greater than π, and as a consequence we cannot apply Proposition 4.1.

We are going to follow another strategy and study the product $X = \mathbb{R}^{n-2} \times C(\mathbb{R}/\alpha\mathbb{Z})$ for $\alpha \geq 2\pi$ from the point of view of *isoperimetric profiles*.

Definition 10. Given a metric-measure space (X, d, μ), the isoperimetric profile of X is the function $I_\mu : \mathbb{R}_+ \to \mathbb{R}$ defined by:

$$I_\mu(v) = \inf\{\mu(\partial E), E \subset X, \mu(E) = v\}.$$

The story of isoperimetric profiles, that is minimizing the boundary which contains a fixed volume (or equivalently, maximizing the volume contained in a given boundary), goes back to the Roman and Greek mythology [VFK06]: Dido, a Phoenician queen escaping from her country after the assassinate of her husband, arrived to the coasts of North Africa and asked to the Berber king Hiarbas the permission to build a town for her people. Hiarbas, convinced of being smarter than Dido, told her to take the region that could be bounded by an oxhide. But Dido cut the oxhide in fine strips and managed to have enough space to lay the foundations of Carthage.

Isoperimetric problems and their relations to other geometric and analytic properties of manifolds, or more in general metric-measure spaces, constitutes a wide, and constantly developing, area of research in mathematics: we cite here as few examples [Ros05a], [Gal88] or [Cha01]. What we are most interested in for our scopes, is the Euclidean isoperimetric inequality.

We say that an Euclidean isoperimetric inequality holds on a metric-measure space X of dimension n if the exists a constant c such that:

$$I_\mu \geq cv^{1-\frac{1}{n}} \tag{4.4}$$

In the Euclidean space $\mathbb{R}^n$ with the standard metric, the constant is given by the isoperimetric quotient c_n of the ball of radius one, and the isoperimetric profile is exactly the function $c_n v^{1-1/n}$. We can reformulate this in terms of an isoperimetric inequality: for any bounded domain $E \subset \mathbb{R}^n$ with C^1-boundary and volume v, denote by E^s the Euclidean ball centred at zero and of volume v. Then we have:

$$\frac{\mathrm{Vol}(\partial E)}{\mathrm{Vol}(E)^{\frac{n-1}{n}}} \geq \frac{\mathrm{Vol}(\partial E^s)}{v^{\frac{n-1}{n}}} \tag{4.5}$$

It is a well-known result that the isoperimetric inequality in $\mathbb{R}^n$ is equivalent to a sharp Sobolev inequality.

Theorem 4.4. *The inequality (4.5) holds for any bounded domain E in $\mathbb{R}^n$, $n > 1$, with C^1-boundary if and only if for there exists a constant C such that for any function f in $W^{1,1}(\mathbb{R}^n)$ we have:*

$$\|f\|_q \leq C\,\|df\|_1\,, \quad q = \frac{n}{n-1}.$$

By knowing the value of c_n (the isoperimetric quotient of the ball of radius one), it is also possible to compute the explicit value for the optimal constant appearing in this last inequality. This leads in turns to the following sharp inequalities for $1 \leq p < n$, see for example [Tal76]:

$$\|f\|_q \leq C_{n,p}\,\|df\|_p\,, \quad q = \frac{np}{n-p} \tag{4.6}$$

In what follows, we are going to show that the isoperimetric profile of the product $X = \mathbb{R}^{n-2} \times C(\mathbb{R}/\alpha\mathbb{Z})$ with the appropriate metric coincides with the isoperimetric profile of $\mathbb{R}^n$. By following the classical argument of G. Talenti, this will give us a sharp Sobolev inequality and consequently the value of the Yamabe constant of X.

4.2.1 Approaching $C(\mathbb{R}/\alpha\mathbb{Z})$ with Cartan-Hadamard manifolds

On the product $X = \mathbb{R}^{n-2} \times C(\mathbb{R}/\alpha\mathbb{Z})$ we consider the metric $\xi + dr^2 + (ar)^d\theta^2$, where ξ is the Euclidean metric and $a = \alpha/2\pi$. We first approximate the cone $C(\mathbb{R}/\alpha\mathbb{Z})$ by Cartan-Hadamard manifolds, that are complete, simply connected Riemannian manifolds with non-negative sectional curvatures. This represents a very simple example of a stratified space appearing as a limit of smooth manifolds. We are going to find a metric h_ε on $\mathbb{R}^2$ which has negative sectional curvature, it is conformal to the Euclidean metric, and which converges to a metric h on $\mathbb{R}^2$ with one conical singularity. This is in turn isometric to $C(\mathbb{R}/\alpha\mathbb{Z})$ endowed with the metric $dr^2 + (ar)^2 d\theta^2$.

Lemma 4.5. *There exists a sequence of metrics h_ε on $\mathbb{R}^2$, conformal to the Euclidean metric, with negative sectional curvature, such that h_ε converges uniformly on any compact domain of $\mathbb{R}^2 \setminus \{0\}$ to the cone metric on $C(\mathbb{R}/\alpha\mathbb{Z})$ with $\alpha > 2\pi$.*

Proof. Consider the following metric on $\mathbb{R}^2$:

$$h_\varepsilon = (\varepsilon^2 + \rho^2)^{a-1}(d\rho^2 + \rho^2 d\theta^2) \tag{4.7}$$

We can compute its sectional curvature κ_ε by applying the formulas for conformal changes of metrics (see for example [Bes08]):

$$\begin{aligned} g =& e^{2f_\varepsilon}(d\rho^2 + \rho^2 d\theta^2), \quad f_\varepsilon = \frac{a-1}{2}\ln(\rho^2+\varepsilon^2) \\ \kappa_\varepsilon =& e^{-2f_\varepsilon}\Delta_g f_\varepsilon = -\frac{2(a-1)\rho}{(\rho^2+\varepsilon^2)^{a+1}}. \end{aligned}$$

Therefore κ_ε is non-positive, since by assumption $a \geq 1$. When ε tends to zero, the conformal factor $(\rho^2+\varepsilon^2)^{a-1}$ converges to $\rho^{2(a-1)}$ pointwise and uniformly in C^∞ on any compact domain. As a consequence $h_\varepsilon(\rho,\theta)$ converges to

$$h(\rho,\theta) = \rho^{2(a-1)}(d\rho^2 + \rho^2 d\theta^2)$$

which is a Riemannian metric on $\mathbb{R}^2 \setminus \{\rho = 0\}$. Now, $\mathbb{R}^2$ endowed with the metric h is a surface with one conical singularity at $(0,0)$, which is isometric to $C(\mathbb{R}/\alpha\mathbb{Z})$ endowed with the metric $dr^2 + (ar)^2 d\theta^2$: it suffices to apply the change of variables $r = \rho^a/a$. □

The interest of Cartan-Hadamard manifolds four us lies in the following conjecture, which is known as the Cartan-Hadamard conjecture or the Aubin conjecture (see for example [Rit05]):

Conjecture 4.6 (Cartan-Hadamard, Aubin)**.** *Let (M^n, g) be a Cartan-Hadamard manifold, whose sectional curvatures satisfy $\kappa \leq c \leq 0$. Then the isoperimetric profile I_M of M^n is bounded from below by the isoperimetric profile I_c of the complete and simply connected space M^n_c, whose sectional curvatures are equal to c.*

This conjecture has been proved in dimension $n = 2, 3, 4$ respectively by A. Weil [Wei26], C. Croke [Cro80] and B. Kleiner [Kle92]. In our particular case, $(\mathbb{R}^2, h_\varepsilon)$ is a Cartan-Hadamard manifold with $c = 0$. As a consequence we have:

Lemma 4.7. *Let h_ε be the metric on $\mathbb{R}^2$ defined in the previous lemma. Then the isoperimetric profile I_{h_ε} of $(\mathbb{R}^2, h_\varepsilon)$ is bounded by below by the isoperimetric profile I_2 of $\mathbb{R}^2$ with the Euclidean metric.*

4.2.2 Isoperimetric profiles

Since we can approximate the cone $C(\mathbb{R}/\alpha\mathbb{Z})$ by Cartan-Hadamard manifolds, X is approximated by the Riemannian product $(\mathbb{R}^{n-2}\times\mathbb{R}^2, \xi+h_\varepsilon)$. We need to get information on the isoperimetric profiles of this latter starting from the bound from below for I_{h_ε}. In order to do that, we recall a result which is known in the literature as Ros Product Theorem and contained in [Ros05a], about the isoperimetric profiles of Riemannian products.

Proposition 4.8 (Ros Product Theorem)**.** *Consider two Riemannian manifolds (M_1, g_1) and (M_2, g_2), $\dim(M_2) = n$. Assume that the isoperimetric profile I_2 of (M_2, g_2) is bounded by below by the isoperimetric profile I_n of $\mathbb{R}^n$. Then the isoperimetric profile of the Riemannian product $(M_1 \times M_2, g_1 + g_2)$ is bounded by below by the one of $(M_1 \times \mathbb{R}^n, g_1 + \xi)$, where ξ is the Euclidean metric.*

The proof consists in defining an appropriate symmetrization for subsets E of $M_1 \times M_2$. Denote for simplicity $g = g_1 + g_2$ and $g_0 = g_1 + \xi$. We consider for $x \in M_1$ the section $E(x) = E \cap (\{x\} \times M_2)$. Then the symmetrization $E^s \subset M_1 \times \mathbb{R}^n$ will be the set defined by:

1. if $E(x) = \emptyset$, then $E^s(x) = \emptyset$;
2. if $E(x) \neq \emptyset$, then $E^s(x) = \{x\} \times B_r$, where B_r is an euclidean ball in $\mathbb{R}^n$ such that $\mathrm{Vol}_\xi(B_r) = \mathrm{Vol}_{g_2}(E(x))$.

By following Proposition 3.6 in [Ros05a], E^s satisfies that $\mathrm{Vol}_{g_0}(E^s) = \mathrm{Vol}_g(E)$ and $\mathrm{Vol}_{g_0}(\partial E^s) \leq \mathrm{Vol}_g(\partial E)$. This is enough to show that if $F \subset M_1 \times M_2$ realizes the infimum in $I_g(v)$, i.e $\mathrm{Vol}_g(F) = v$ and $\mathrm{Vol}_g(\partial F) = I_g(v)$, then its symmetrization F^s satisfies $\mathrm{Vol}_{g_0}(F^s) = v$ and

$$I_{g_0}(v) \leq \mathrm{Vol}_{g_0}(\partial F^s) \leq \mathrm{Vol}_g(\partial F)$$

As a consequence, $I_g(v) \geq I_{g_0}(v)$ for any $v > 0$.

Proposition 4.9. *Let $X = \mathbb{R}^{n-2} \times C(\mathbb{R}/\alpha\mathbb{Z})$ and denote by g the metric $\xi + h$. Let I_g be its isoperimetric profile. Then I_g is coincides with the isoperimetric profile I_n of $\mathbb{R}^n$ with the Euclidean metric.*

Proof. We will show first that I_g is bounded by below by I_n. Consider $\mathbb{R}^{n-2} \times \mathbb{R}^2$ endowed with the metric $g_\varepsilon = \xi + h_\varepsilon$, and denote by I_ε the isometric profile with respect to this metric. Thanks to Lemma 4.7 and to Ros Product theorem we deduce that I_ε is bounded by below by the isoperimetric profile of $\mathbb{R}^{n-2} \times \mathbb{R}^2$ with the euclidean metric, that is I_n.
Therefore we have the following isoperimetric inequality for any bounded domain $E \subset X$, $\mathrm{Vol}_{g_\varepsilon}(\Omega) = v$, with smooth boundary ∂E:

$$\frac{\mathrm{Vol}_{g_\varepsilon}(\partial E)}{\mathrm{Vol}_{g_\varepsilon}(E)^{1-\frac{1}{n}}} \geq \frac{I_\varepsilon(v)}{v^{1-\frac{1}{n}}} \geq c_n \tag{4.8}$$

where c_n is the optimal constant appearing in the isoperimetric inequality in $\mathbb{R}^n$. When we pass to the limit as ε tends to zero, the volumes of both E and ∂E with respect to g_ε converge to the volumes with respect to g.

In fact, if we denote by dx the n-dimensional Lebesgue measure on $\mathbb{R}^n$ and by $d\sigma$ the volume element induced on ∂E by the Euclidean metric, we have for the volume of E:

$$\lim_{\varepsilon\to 0} \mathrm{Vol}_{g_\varepsilon}(E) = \int_E (\rho^2+\varepsilon^2)^{(a-1)} dx = \mathrm{Vol}_g(E)$$

since $(\rho^2+\varepsilon^2)^{(a-1)}$ converges to $\rho^{2(a-1)}$ on any bounded domain. As for the volume of ∂E we get:

$$\begin{aligned}
\lim_{\varepsilon\to 0} \mathrm{Vol}_{g_\varepsilon}(\partial E) &= \lim_{\varepsilon\to 0} \int_{\partial E} (\rho^2+\varepsilon^2)^{(a-1)} d\sigma \\
&= \lim_{\varepsilon\to 0} \int_{\partial E\setminus \mathbb{R}^{n-2}\times\{0\}} (\rho^2+\varepsilon^2)^{(a-1)} d\sigma \\
&= \int_{\partial E\setminus \mathbb{R}^{n-2}\times\{0\}} \rho^{2(a-1)} d\sigma \\
&= \int_{\partial E} \rho^{2(a-1)} d\sigma = \mathrm{Vol}_g(\partial E).
\end{aligned}$$

where we used again the convergence of the conformal factor and the fact that $\mathbb{R}^{n-2}\times\{0\}$ has zero $(n-1)$-dimensional Lebesgue measure.

Therefore when we pass to the limit as ε goes to zero in 4.8 we obtain:

$$\frac{\mathrm{Vol}_g(\partial E)}{\mathrm{Vol}_g(E)^{1-\frac{1}{n}}} \geq c_n \tag{4.9}$$

Observe that $\mathbb{R}^{n-2}\times C^1(S_a)$ contains Euclidean balls: they are the geodesic balls $\mathbb{B}^n$ not intersecting the singular set $\mathbb{R}^{n-2}\times\{0\}$. They attains the constant c_n, so that for any $v>0$ the infimum defining $I_g(v)$ is attained by the Euclidean geodesic ball of volume v, i.e. $I(v) = c_n v^{1-\frac{1}{n}}$. As a consequence, the isoperimetric profile I_g coincides with I_n. □

4.2.3 Yamabe constant of $\mathbb{R}^{n-2}\times C(\mathbb{R}/\alpha\mathbb{Z})$

We have found an optimal constant for the isoperimetric inequality (4.9) with respect to a metric $g = \xi + h$ on $X = \mathbb{R}^{n-2}\times C(\mathbb{R}/\alpha\mathbb{Z})$. Such metric is isometric to $\xi + dr^2 + (ar)^2 d\theta^2$ on X, so they obviously define the same conformal class. As a consequence, we can compute the Yamabe constant of $\mathbb{R}^{n-2}\times C(\mathbb{R}/\alpha\mathbb{Z})$, as the following proposition shows.

Proposition 4.10. *The Yamabe constant of $X = \mathbb{R}^{n-2}\times C(\mathbb{R}/\alpha\mathbb{Z})$, $a>1$, is equal to the Yamabe constant Y_n of the standard sphere of dimension n.*

Proof. As we recalled above, in the Euclidean space $\mathbb{R}^n$, the existence of the isoperimetric inequality leads to the existence of a sharp Sobolev inequality: for any $1<p<n$ and for any $f\in W^{1,p}(\mathbb{R}^n)$:

$$\|f\|_q \leq C_{n,p}\|df\|_p, \quad q = \frac{np}{n-p}. \tag{4.10}$$

The constant $C_{n,p}$ is optimal in the sense that it is such that:

$$C_{n,p}^{-1} = \inf_{\substack{f \in W^{1,p}(\mathbb{R}^n) \\ f \neq 0}} \frac{\|df\|_p}{\|f\|_q}. \tag{4.11}$$

We briefly recall the ideas of the proof given by G. Talenti in [Tal76]. For any Lipschitz function u we can define the symmetrization u^* in the following way: for any $t \in \mathbb{R}$, the level set $E_t^* = \{x \in \mathbb{R}^n : u^*(x) > t\}$ of u^* are Euclidean n-balls having the same volume as the level set E_t of u. Then u is spherically symmetric and Lipschitz. It is possible to show that this kind of symmetrization makes the ratio (4.11) decrease: from Lemma 1 in [Tal76] we have that for any $1 < p < n$

$$\|u\|_p = \|u^*\|_p \qquad \text{and} \qquad \|du\|_p \geq \|du^*\|_p.$$

The first equality is trivial. The second inequality is deduced by using isoperimetric inequality and coarea formula, which relates the integral of $|du|$ with the $(n-1)$-measure of the boundaries ∂E_t of level sets.
As a consequence, the infimum in (4.11) is attained by spherically symmetric functions. Classical argument of calculus of variations allows to prove that there exists a minimizer. Moreover, G. Talenti exhibits a family of functions attaining $C_{n,p}$ and gives its exact value.
When $p = 2$, $(C_{n,2})^{-2}$ coincides with the Yamabe constant Y_n of the n-dimensional sphere. This is shown by pulling back the functions attaining $C_{n,2}$ from $\mathbb{R}^n$ to the sphere $\mathbb{S}^n$ without the north pole.
In our case, $X = \mathbb{R}^{n-2} \times C(\mathbb{S}^1_a)$ is flat and satisfies the Euclidean isoperimetric inequality (4.9). We can then repeat the same argument as Talenti to deduce that the inequality (4.10) holds on X with the same optimal constant $C_n, 2$ as in $\mathbb{R}^n$. Furthermore, by definition of the Yamabe constant, and since $S_g = 0$, we have:

$$Y(X,[g]) = \inf_{\substack{u \in W^{1,2}(X) \\ u \neq 0}} \frac{\int_X |du|^2 dv_g}{\|u\|^2_{\frac{2n}{n-2}}}.$$

so that $Y(X,[g])$ is equal to $(C_{n,2})^{-2}$. We have then proved $Y(X,[g]) = Y_n$. $\square$

As a consequence of the previous results, we are able to compute the local Yamabe constant of any stratified space whose links are endowed with an Einstein metric:

Proposition 4.11. *Let (X,g) be a stratified space with strata X^j, $j = 1 \dots n$ and links Z_j of dimension d_j endowed with an Einstein metric k_j. Then the local Yamabe constant of (X,g) is equal to:*

$$Y_\ell(X) = \inf_{j=1,\dots n} \left\{ Y_n, \left(\frac{\mathrm{Vol}_{k_j}(Z_j)}{\mathrm{Vol}(\mathbb{S}^{d_j})} \right)^{\frac{2}{n}} Y_n \right\}.$$

4.3 A second approach

Consider a compact smooth manifold (M^n, g), of dimension n greater or equal than 3, with an Einstein metric $Ric_g = (n-1)g$. A result of M. Obata contained in [Oba72] states that if (M^n, g) admits a conformal metric $\tilde{g}$ with constant scalar curvature, then (M^n, g) is conformally equivalent to the standard sphere of dimension n. In particular, an Einstein metric attains the Yamabe constant. The proof of this result is based on the existence of a conformal vector field X on (M^n, g), that is a vector field X such that the Lie derivative $\mathcal{L}_X g$ of the metric along X is proportional to the metric.

We are going to show a similar result on admissible stratified spaces which admit a Yamabe minimizer. We divide the proof into two steps: first we prove the existence of a conformal vector field on an admissible stratified space, by following an argument of J. Viaclovsky (see proof of Theorem 1.3 in [Via10]), then we deduce that an Einstein metric is a Yamabe metric.

Theorem 4.12. *Let (X, g) be an admissible Einstein stratified space of dimension n. Assume that there exists a metric $\tilde{g}$ in the conformal class of g with constant scalar curvature. Then $\tilde{g}$ is an Einstein metric and there exists a function ϕ satisfying:*

$$\nabla d\phi = -\frac{\Delta_g \phi}{n} g. \tag{4.12}$$

In particular, the vector field $X = d\phi$ is a conformal vector field such that $\mathcal{L}_X g = -2\phi g$.

Before proving this theorem we give two lemmas that allows us to deduce that, under the previous hypothesis, a Yamabe minimizer belongs to the Sobolev space $W^{2,2}(X)$ and its gradient is bounded.

Lemma 4.13. *Let (X, g) be an Einstein stratified space of dimension n, such that the stratum of codimension 2, if it exists, has link of diameter smaller than π. Assume that there exists a solution $u \in W^{1,2}(X) \cap L^\infty(X)$ to the Yamabe equation:*

$$\Delta_g u + a_n S_g u = a_n S_g u^{\frac{n+2}{n-2}}. \tag{4.13}$$

Then for any $\varepsilon > 0$ we have the following control of the gradient away from an ε-tubular neighbourhood of the singular set Σ:

$$\|du\|_{(X \setminus \Sigma^\varepsilon)} \leq C\sqrt{|\ln \varepsilon|}. \tag{4.14}$$

where C is a positive constant not depending on ε.

In fact, since S_g is equal to a constant and the Ricci tensor is in particular bounded by below, it suffices to remark that

$$F(x) = (x^{\frac{4}{n-2}} - 1) a_n S_g x.$$

is a locally Lipschitz function, and then apply Corollary 2.11. As we did in the proof of Theorem 2.5, we can deduce from the previous lemma and from the Yamabe equation that the gradient of a Yamabe minimizer belongs to $W^{1,2}(X) \cap L^\infty(X)$.

Lemma 4.14. *Let (X, g) be an admissible stratified space of dimension n with Einstein metric. Then the gradient $|\nabla u|$ of a solution u to the Yamabe equation belongs to $L^\infty(X)$.*

Proof. From the previous lemma we know that on Ω we have:

$$\frac{1}{2}\Delta_g(|\nabla u|^2) = (\nabla^*\nabla du, du) - |\nabla du|^2 \leq c_1|\nabla u|^2 - |\nabla du|^2.$$

Denote by m the codimension of the singular set. If $m > 2$, consider the cut-off function ρ_ε defined in the proof of Theorem 2.5. If $m = 2$ consider the function f_ε defined in Lemma 1.6. Then multiply the previous inequality by ρ_ε if $m > 2$, by f_ε otherwise. If we integrate by parts we obtain:

$$\frac{1}{2}\int_X (\Delta_g\rho_\varepsilon)|\nabla u|^2 dv_g \leq c_1\int_X \rho_\varepsilon|\nabla u|^2 dv_g - \int_X \rho_\varepsilon|\nabla du|^2 dv_g. \tag{4.15}$$

Thanks to the fact that ∇u satisfies (4.14), like any eigenfunction of the Laplacian, we know from Theorem 2.5 that ρ_ε is chosen is such a way that the left-hand side of (4.15) tends to zero as ε tends to 0: as a consequence, the norm in $L^2(X)$ of ∇du is bounded by the one of $|\nabla u|$, which is finite. This means that $\nabla|\nabla u|$ belongs to $L^2(X)$, and $|\nabla u|$ to $W^{1,2}(X)$. If $m = 2$, and we have f_ε instead of ρ_ε, we know that the left-hand side of (4.15) is bounded by a constant independent of ε, and we get to the same conclusion. Therefore, we can apply the same argument as in the claim contained in the proof of Theorem 2.5 in order to deduce that $|\nabla u|$ satisfies the weak inequality $\Delta_g|\nabla u| \leq c_1|\nabla u|$ on the whole X. As a consequence, the Moser iteration technique implies that $|\nabla u|$ is bounded on X. □

We are now in position to prove Theorem 4.12.

Proof of Theorem 4.12. By assumption, there exists a conformal metric $\tilde{g} \in [g]$ with constant scalar curvature: we can assume without loss of generality that $S_{\tilde{g}} = S_g$. If ϕ is a positive function such that $\tilde{g}$ has the form $\tilde{g} = \phi^{-2}g$, then there must exist a function $u \in W^{1,2}(X) \cap L^\infty(X)$ solving the Yamabe equation:

$$\Delta_g u + \frac{n(n-2)}{4}u = \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}}$$

and such that $\phi = u^{-\frac{2}{n-2}}$. We know from Theorem 1.12 in [ACM14] that u, and thus ϕ, is positive and bounded. By the two previous Lemmas we also have that the gradient of u, and therefore the gradient of ϕ, belongs to $L^\infty(X)$.

Consider the traceless Ricci tensor $E_{\tilde{g}}$ and recall that $\tilde{g}$ is an Einstein metric if and only if $E_{\tilde{g}}$ vanishes: our goal is to show that this is the case. The transformation law for the traceless Ricci tensor under a conformal change (see for example [Bes08]) gives us the following formula for $E_{\tilde{g}}$:

$$E_{\tilde{g}} = E_g + (n-2)\phi^{-1}\Big(\nabla^2\phi + \frac{\Delta_g\phi}{n}g\Big)$$

where the covariant derivatives are taken with respect to g. Since by assumption g is an Einstein metric, $E_g = 0$. Then consider the following integral:

$$I_\varepsilon = \int_X \rho_\varepsilon \phi |E_{\tilde{g}}|_g^2 dv_g$$

where ρ_ε is chosen like in the proof of Theorem 2.5 depending on the codimension of the singular set. If we show that I_ε tends to zero as ε goes to zero, then the norm of $E_{\tilde{g}}$ must vanish: as a consequence we will obtain that $\tilde{g}$ is an Einstein metric and that its conformal factor ϕ satisfies (4.12). Let us rewrite I_ε in the appropriate form:

$$\begin{aligned}
I_\varepsilon &= \int_X \rho_\varepsilon \phi \left(E_{\tilde{g}}, (n-2)\phi^{-1} \left(\nabla d\phi - \frac{\Delta_g \phi}{n} g \right) \right)_g dv_g \\
&= (n-2) \int_X \rho_\varepsilon \left(E_{\tilde{g}}, \nabla d\phi - \frac{\Delta_g \phi}{n} g \right)_g dv_g \\
&= (n-2) \int_X \rho_\varepsilon \left(E_{\tilde{g}}, \nabla d\phi \right)_g dv_g
\end{aligned}$$

In the last equality we used that the Laplacian of ϕ is the trace of the Hessian $\nabla d\phi$. Then we integrate by parts:

$$\int_X \rho_\varepsilon \left(E_{\tilde{g}}, \nabla d\phi \right)_g dv_g = \int_X (E_{\tilde{g}}^{ij} \nabla_j \rho_\varepsilon \nabla_i \phi + \rho_\varepsilon \nabla_j E_{\tilde{g}}^{ij} \nabla_i \phi) dv_g.$$

Since the scalar curvature of $\tilde{g}$ is constant, by the Bianchi identity (see also [BE87]), which holds on the regular set of X, the second term of this integral is equal to zero. The first one leads to:

$$I_\varepsilon = (n-2)^2 \int_X \phi^{-1} \left(\nabla d\phi(\nabla \rho_\varepsilon, \nabla \phi) - \frac{\Delta_g \phi}{n} (\nabla \rho_\varepsilon, \nabla \phi)_g \right) dv_g. \tag{4.16}$$

Observe that ϕ^{-1} is positive and bounded, because the solution u to the Yamabe equation is positive and bounded thanks to Theorem 1.12 in [ACM14]. We claim that the Laplacian of ϕ is bounded as well. In fact, if we denote $p = -\frac{2}{n-2}$ we have:

$$\Delta_g \phi = p u^{p-1} \left(\Delta_g u - (p-1) \frac{|\nabla u|^2}{u} \right).$$

As we recalled above, the function u is bounded and positive, then its Laplacian $\Delta_g u$ is bounded, since it is equal to:

$$\Delta_g u = \frac{n(n-2)}{4} u (u^{\frac{4}{n-2}} - 1).$$

Moreover, by the previous Lemmas the gradient $|\nabla u|$ belongs to $L^\infty(X)$, so that the same holds for $\Delta_g \phi$. Therefore, if we consider the last term in (4.16), we know from the proof of Theorem 2.5 that ρ_ε is chosen in such a way that the integral of $(\nabla \rho_\varepsilon, \nabla u)$ goes to zero as ε tends to zero.

As for the first term in (4.16), we can integrate by parts and obtain:

$$\int_X \nabla d\phi(\nabla\rho_\varepsilon, \nabla\phi) dv_g = \frac{1}{2}\int_X \rho_\varepsilon \Delta_g |\nabla\phi|^2 dv_g = \frac{1}{2}\int_X (\Delta_g \rho_\varepsilon)|\nabla\phi|^2 dv_g.$$

We have shown in the proof of Theorem 2.5 that ρ_ε is such that this last term tends to zero as ε goes to zero as well.

As a consequence, we have shown that I_ε tends to zero as ε goes to zero. Therefore we obtain that the norm of the traceless Ricci tensor $E_{\tilde{g}}$ is equal to zero, the metric $\tilde{g}$ is an Einstein metric and the function ϕ satisfies (4.12), as we wished. ☐

A scalar function solving the equation (4.12) is called in the literature a concircular scalar field. The existence of a concircular scalar field or of a conformal vector field on a compact, or complete, smooth manifold can lead to various consequences. For example, Y. Tashiro in [Tas65] classified complete manifolds possessing a concircular scalar field. See also Sections 2 and 3 of [Mon99] for a brief but complete presentation of some known results about the subject.

In our case, the previous theorem leads to the following:

Corollary 4.15. *Let (X, g) be an admissible Einstein stratified space of dimension n admitting a Yamabe minimizer*

$$\tilde{g} = \phi^{-2/(n-2)} g$$

Assume that ϕ is not a constant function. Then the Einstein metric g attains the Yamabe constant, which is consequently equal to

$$Y(X, [g]) = \frac{n(n-2)}{4} \mathrm{Vol}_g(X)^{\frac{2}{n}}.$$

Proof. We have proven in the previous theorem that any metric with constant scalar curvature in the conformal class of g is an Einstein metric and it is determined by a positive solution of (4.12). Up to multiplying by a constant, a positive solution of (4.12) is given by

$$\phi_t = (1-t)\phi + t$$

for some $t \in [0, 1)$. Let us denote:

$$u_t = \phi_t^{-\frac{2n}{n-2}}.$$

the corresponding solution to the Yamabe equation. The metric $g_t = \phi_t^{-2} g$ is still an Einstein metric in the conformal class of g and has the same scalar curvature as g.

We want to show that the volume of X with respect to the metric g_t is constant in t: this means that it is constant among the metrics with constant scalar curvature equal to $n(n-1)$. In this way, the ratio

$$Q(\tilde{g}) = \frac{a_n \int_X \mathrm{Scal}_{\tilde{g}} dV_{\tilde{g}}}{\mathrm{Vol}_{\tilde{g}}(X)^{1-\frac{2}{n}}}$$

does not decrease in the set of conformal metrics with constant scalar curvature. As a consequence, the Yamabe constant of (X, g) will be attained by g and it is equal to:

$$Y(X, [g]) = \frac{n(n-2)}{4} \mathrm{Vol}_g(X)^{\frac{2}{n}}.$$

The volume of X with respect to g_t is given by the formula

$$\mathrm{Vol}_{g_t}(X) = \int_X u_t^{\frac{2n}{n-2}} dv_g = \int_X dv_{g_t}.$$

where we denote with dv_{g_t} the volume element with respect to g_t. If we differentiate with respect to t we get

$$\frac{\mathrm{d}}{\mathrm{dt}} \mathrm{Vol}_{g_t}(X) = \frac{2n}{n-2} \int_X u_t^{\frac{n+2}{n-2}} \dot{u}_t dv_g = \frac{2n}{n-2} \int_X \frac{\dot{u}_t}{u_t} dv_{g_t}. \tag{4.17}$$

We are going to show that this integral is equal to zero. If we set

$$v_h = \frac{u_{t+h}}{u_t}.$$
$$g_h = v_h^{\frac{4}{n-2}} g_t = u_{t+h}^{\frac{4}{n-2}} g.$$

then v_h satisfies the Yamabe equation with respect to g_t:

$$\Delta_{g_t} v_h + \frac{n(n-2)}{4} v_h = \frac{n(n-2)}{4} v_h^{\frac{n+2}{n-2}}.$$

By deriving this equality with respect to h we obtain

$$\Delta_{g_t} \dot{v}_h + \frac{n(n-2)}{4} \dot{v}_h = \frac{n(n+2)}{4} v_h^{\frac{4}{n-2}} \dot{v}_h.$$

and when $h = 0$ we have as a consequence $\Delta_{g_t} \dot{v}_0 = n\dot{v}_0$, that is v_0 is an eigenfunction relative to the first eigenvalue n of Δ_{g_t}. Any eigenfunction relative to the first eigenvalue has mean equal to zero over X, so that we have:

$$\int_X \dot{v}_0 dV_{g_t} = 0.$$

But by definition $\dot{v}_0$ is equal to $\dfrac{\dot{u}_t}{u_t}$. Recalling (4.17) we have obtained that the volume of X is constant with respect to t: this implies that the Einstein metric g attains the Yamabe constant, as we wished. □

The issue with this last result is that it can be applied only provided that we know that a Yamabe minimizer exists, which is a non trivial assumption. However, we can show that this hypothesis is satisfied in some cases. In order to do that, we are going to consider *almost homogeneous* manifolds.

4.3.1 Yamabe minimizer on almost homogeneous manifolds

In this section we introduce a result due to K. Akutagawa and proven by N. Grosse which states the existence of a Yamabe minimizer on a class of complete open manifolds, *almost homogeneous* manifolds (see Theorem 13 in [Gro13]). N. Grosse's proof is based on weighted Sobolev embeddings: we present here an alternative argument inspired by [AB03] and [ACM14], which mainly relies on Moser's iteration technique.

We start by giving a few results which hold on complete open manifolds. The first one is due to E. Hebey and M. Vaugon [HV95] (see also Theorem 7.2 in [Heb99]): it states the existence of a Sobolev inequality with an explicit optimal constant.

Theorem 4.16. *Let (M^n, g) be a smooth, complete, Riemannian manifold with $n \geq 3$. Suppose that the Riemann curvature Rm_g is such that*

$$|Rm_g| \leq \Lambda_1 \quad |\nabla Rm_g| \leq \Lambda_2$$

for some non-negative constants Λ_1, Λ_2, and its injectivity radius is such that

$$\mathrm{inj}_g \geq i > 0.$$

Then there exists $B = B(n, i, \Lambda_1, \Lambda_2)$ such that for any $u \in W^{1,2}(M)$ we have:

$$\Big(\int_M |u|^{\frac{2n}{n-2}} dv_g\Big)^{\frac{n-2}{n}} \leq \frac{1}{Y_n}\int_M |du|^2 dvg + B\int_M u^2 dv_g. \tag{4.18}$$

where Y_n is the Yamabe constant of the standard sphere of dimension n.

From this theorem we can easily deduce the following:

Lemma 4.17. *Under the same hypothesis as in the previous theorem, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all x in M^n and u belonging to $C^\infty(B(x,\delta))$ we have:*

$$\Big(Y_n - \varepsilon\Big)\Big(\int_{B(x,\delta)} u^{\frac{2n}{n-2}} dv_g\Big)^{\frac{n-2}{n}} \leq \int_{B(x,\delta)} |du|^2 dv_g. \tag{4.19}$$

Proof. The proof simply depends on the fact that the volume of small balls $B(x,\delta)$ is bounded by a constant times δ^n (see the classical theorem 3.98 in [GHL04]). Consider a function u in $C^\infty(B(x,\delta))$: by the Sobolev inequality (4.18) and the Hölder inequality we obtain

$$\begin{aligned}
Y_n\Big(\int_{B(x,\delta)} u^{\frac{2n}{n-2}} dv_g\Big)^{\frac{n-2}{n}} &\leq \int_{B(x,\delta)} |du|^2 dv_g + BY_n\int_{B(x,\delta)} u^2 dv_g \\
&\leq \int_{B(x,\delta)} |du|^2 dv_g + BY_n \,\mathrm{Vol}_g(B(x,\delta))^{\frac{2}{n}}\Big(\int_{B(x,\delta)} u^{\frac{2n}{n-2}} dv_g\Big)^{\frac{n-2}{n}}.
\end{aligned}$$

and since there exists a constant C independent of x such that $\mathrm{Vol}_g(B(x,\delta) \leq C\delta^n$, we get for some positive C_1:

$$Y_n\Big(\int_{B(x,\delta)} u^{\frac{2n}{n-2}} dv_g\Big)^{\frac{n-2}{n}} \leq \int_{B(x,\delta)} |du|^2 dv_g + C_1\delta^2\Big(\int_{B(x,\delta)} u^{\frac{2n}{n-2}} dv_g\Big)^{\frac{n-2}{n}}.$$

Then by taking $\delta < \sqrt{\varepsilon/C'}$ we get the desired inequality. $\square$

The previous two forms of the Sobolev inequality suffice to apply the Moser iteration technique to a solution of the Schrödinger equation $\Delta_g u - Vu = 0$, provided that we can control the potential V in the appropriate way. The following theorem will be our main tool to give a new proof of Theorem 13 in [Gro13].

Theorem 4.18. *Let (M^n, g) be a complete Riemannian manifold satisfying the hypothesis of Theorem 4.16. Assume that V belongs to $L^q_{loc}(M)$ for some $q > \frac{n}{2}$ and that there exist $\delta_0 > 0$ and a positive constant C_0 such that for any $x \in M$ we have*

$$\Big(\int_{B(x,\delta_0)} |V|^q dv_g\Big)^{\frac{1}{q}} \leq C_0 \tag{4.20}$$

Let u be in $W^{1,2}(M)$ and assume it satisfies $\Delta u - Vu \leq 0$ in the weak sense. Then u is bounded on M.

Proof. What we are going to show is that there exists an appropriate radius r such that the function u is bounded on each ball $B(x, r)$ by a constant which is independent of x and r. This implies clearly that u is bounded. In order to do that we will introduce a cut-off function supported on a ball like in the proof of Lemma 1.16, and test the Sobolev inequality on powers of u to increase the exponent q such that u belongs to $L^q(X)$. We need to be sure that we can iterate this procedure on powers of u: consider then a convex non-decreasing function f. We claim that

$$\Delta f(u) \leq V f'(u) u \tag{4.21}$$

Let $\{P_t, t > 0\}$ be the semi-group of bounded operators associated to the self-adjoint operator Δ. For any test function $\varphi \in C_c^\infty(M)$ we have

$$\int_M f(u)\Delta\varphi = \int_M \lim_{t\to 0} \frac{f(u) - P_t f(u)}{t}\varphi$$

Since f is convex, we can apply Jensen's inequality $P_t(f(u)) \geq f(P_t(u))$ and obtain

$$\int_M f(u)\Delta\varphi \leq \int_M \lim_{t\to 0} \frac{f(u) - f(P_t u)}{t}\varphi$$

Use again convexity and the fact that f' is non-negative in order to obtain that

$$f(u) - f(P_t u) \leq f'(u)(u - P_t u)$$

and then:

$$\int_M f(u)\Delta\varphi \leq \int_M f'(u) \lim_{t\to 0} \frac{u - P_t u}{t}\varphi = \int_M f'(u)\Delta u\varphi \leq \int_M f'(u) V u\varphi$$

which proves our claim.
Consider any point $x \in M$ and $0 < r < R$. Let φ be a smooth cut-off function such

that φ has support contained in the ball $B(x,R)$, φ equals one on the ball $B(x,r)$ and its gradient satisfies:

$$|d\varphi| \leq \frac{2}{R-r} \quad \text{on } B(x,R)\setminus B(x,r).$$

Let $\alpha \geq 1$ and consider φu^α. By the integration by parts formula and (4.21) we get

$$\int_M |d(\varphi u^\alpha)|^2 dv_g = \int_M |d\varphi|^2 u^{2\alpha} + \varphi^2(\Delta u^\alpha)u^\alpha \leq \int_M |d\varphi|^2 u^{2\alpha} + \alpha\varphi^2 V u^{2\alpha}$$

Fix a small positive $\varepsilon < 1$ and the corresponding δ in the inequality (4.19). We can choose R to be smaller than the minimum between δ_0 and δ. We can then apply Lemma 4.17 to φu^α and get:

$$\Big(Y_n - \varepsilon\Big)\Big(\int_M (\varphi u^\alpha)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \leq \Big(\int_M |d\varphi|^2 u^{2\alpha} + \alpha\varphi^2 V u^{2\alpha}\Big).$$

Let us denote by $A(\varepsilon) = (Y_n - \varepsilon)^{-1}$. We can use Hölder's inequality with the exponent q on both of the terms; then, by recalling the hypothesis (4.20) on V, we obtain

$$\Big(\int_{B(x,r)} u^{\frac{2\alpha n}{n-2}}\Big)^{\frac{n-2}{n}} \leq A(\varepsilon)\Big(\frac{2}{(R-r)^2} + \alpha C_0\Big)\Big(\int_{B(x,R)} u^{\frac{2\alpha q}{q-1}}\Big)^{\frac{q-1}{q}}$$

Since we have chosen a small radius $R < 1$, and since α is bigger than one, we can collect the constants and write for some positive C_1

$$\Big(\int_{B(x,r)} u^{\frac{2\alpha n}{n-2}}\Big)^{\frac{n-2}{n}} \leq \frac{C_1\alpha}{(R-r)^2}\Big(\int_{B(x,R)} u^{\frac{2\alpha q}{q-1}}\Big)^{\frac{q-1}{q}} \tag{4.22}$$

Observe that, as a consequence of the assumption $q > \frac{n}{2}$, the exponent $\frac{2q}{q-1}$ is strictly less than the critical exponent in Sobolev embedding $\frac{2n}{n-2}$. We can choose α sufficiently close to 1, so that

$$\frac{2\alpha q}{q-1} < \frac{2n}{n-2}$$

Then by Sobolev embeddings, the right-hand side of (4.22) is finite. Let us define

$$k = \frac{n}{n-2}\cdot\frac{q-1}{q} > 1, \quad p = \frac{2q}{q-1}$$

Then the inequality (4.22) can be rewritten in the following form:

$$||u||_{L^{\alpha kp}(B(x,r))} \leq \Big(\frac{C_1\alpha}{(R-r)^2}\Big)^{\frac{1}{2\alpha}} ||u||_{L^{\alpha p}(B(x,R))} \tag{4.23}$$

As in the proof of Lemma 1.16 we need to iterate this last equality by keeping in account that we have two different radii in the left and in the right-hand side. Consider then the sequence of radii

$$R_j = \Big(\frac{1}{2} + 2^{-(j+2)}\Big)R$$

$$r_j = \Big(\frac{1}{2} + 2^{-(j+3)}\Big)R$$

so that $R_j = r_{j-1}$; define $\alpha_j = k^j\alpha$. Then we can apply iteratively (4.23) with α_j instead of α: for $j = N$ we obtain

$$||u||_{L^{\alpha k^N p}(B(x,r_N))} \leq \prod_{j=0}^{N-1} (2^{2(j+3)}C_1 k^j \alpha)^{\frac{1}{2k^j\alpha}} ||u||_{L^{\alpha p}(B(x,R_1))}$$

The constant that appears in the right-hand side is bounded independently of N, as in Lemma 1.16. Besides, the Sobolev embedding $W^{1,2} \hookrightarrow L^{\frac{2n}{n-2}}$ implies that the norm $||u||_{L^{\alpha p}(B(x,R_1))}$ is bounded by the norm of u in $W^{1,2}(M)$: then we can pass to the limit as N goes to infinity and obtain for any $x \in M$

$$||u||_{L^\infty(B(x,\frac{R}{2}))} \leq C||u||_{W^{1,2}(M)}$$

Since R is independent of the point x, u is in fact bounded on M. Observe that, as we did in Chapter 1 while recalling the proof of Proposition 1.12, we can also deduce that u is bounded by its norm in $L^2(M)$. □

We can now introduce the notion of almost homogeneous manifold.

Definition 11. Let (M^n, g) be a complete open manifold of dimension $n \geq 3$. We say that (M^n, g) is an *almost homogeneous* manifold if there exists compact subset $K \subset M$ such that for any $x \in M$ there exists an isometry $\gamma \in \mathrm{Isom}(M)$ which sends x in K.

Clearly a homogeneous manifold (M^n, g) satisfies the previous definition, with K being equal to any of its points. In particular, the product $\mathbb{H}^{n-d} \times Z^d$, for a compact smooth manifold Z^d, is included in the definition: we can take K as $\{x_0\} \times Z$ for a point x_0 in $\mathbb{H}^{n-d}$. The subgroup Γ defined by:

$$\Gamma = \Big\{\gamma \times \mathrm{id}_Z, \gamma \in \mathrm{Isom}(\mathbb{H}^{n-d})\Big\}.$$

is such that

$$\mathbb{H}^{n-d} \times Z^d = \bigcup_{\gamma\in\Gamma} \gamma(K).$$

and thus $\mathbb{H}^{n-d} \times Z^d$ satisfies the previous definition as well.

Observe that an almost homogeneous manifold has positive injectivity radius and its Riemann curvature and its derivatives are bounded: therefore, we can apply Theorem 4.18.

The Yamabe constant of (M^n, g) is defined as usual as the infimum of the Yamabe functional Q_g over smooth positive functions with compact support in M. Explicitly we have:

$$Y(M,[g]) = \inf_{\substack{u\in C_0^\infty(M),\\ u\neq 0}} \frac{E(u)}{||u||^2_{\frac{2n}{n-2}}} = \inf_{\substack{u\in C_0^\infty(M),\\ u\neq 0}} \frac{\displaystyle\int_M (|du|^2 + a_n S_g u^2) dV_g}{\Big(\displaystyle\int_M u^{\frac{2n}{n-2}} dV_g\Big)^{\frac{n-2}{n}}}.$$

As we observed in the previous Chapter, Aubin's inequality 3.1 holds for any smooth complete Riemannian manifold and in particular it is true for an almost homogeneous manifold of dimension larger than 3.

It is possible to prove that the Yamabe constant of an almost homogeneous manifold is either non-negative or not bounded by below:

Lemma 4.19. *Let (M^n, g) be an almost homogeneous manifold and assume that its Yamabe constant is negative. Then $Y(M, [g]) = -\infty$.*

This lemma can be proven by combining two results contained in [Gro13] and in [GN14]. Consider an open manifold (M^n, g) and a compact exhaustion $\{K_i\}_{i\in\mathbb{N}}$ of M: the Yamabe constant at infinity of M is defined by

$$Y(\bar{M}, [g]) = \lim_{i\to+\infty} Y(M \setminus K_i, [g]).$$

Theorem 1.7(2) in [GN14] states that if the Yamabe constant at infinity of an open manifold is negative, then it must be equal to $-\infty$. Remark 14 in [Gro13] shows that the Yamabe constant of an almost homogeneous manifold coincides with its Yamabe constant at infinity, and this implies the above lemma. For the sake of completeness, we also give a proof not explicitly involving the Yamabe constant at infinity.

Proof. : Assume that there exists $\varepsilon > 0$ such that $Y(M, [g]) \leq -\varepsilon$. Then by definition of the Yamabe constant, there must exist a compactly supported smooth function u s.t. $Q_g(u) \leq -\varepsilon$. We can assume that the support of u is contained in a ball of radius R and center p_0 in K, and that $K \subset B(p_0, R)$. For each integer i, consider a point $p_i \in \partial B(p_0, 5iR)$. Then $\{p_i\}_{i\in\mathbb{N}}$ is a sequence of points such that for any $i \neq j$, $d_g(p_i, p_j) \geq 5R$. We can choose an isometry γ_i such that $d_g(\gamma_i(p_0), p_i) \leq R$ (we are saying that there exists an isometry that sends p_0 in $B(p_i, R)$). For $i \leq j$, by triangular inequality, γ_i and γ_j satisfy the following inequality:

$$d_g(\gamma_i(p_0), \gamma_j(p_0)) \geq d_g(p_i, p_j) - d_g(\gamma_i(p_0), p_i) - d_g(\gamma_j(p_0), p_j) \geq 3R$$

Consider $u_i = u \circ \gamma_i$. Then u_i is supported on $B(\gamma_i(p_0), R)$, and thanks to the previous inequality for $i \neq j$ the functions u_i and u_j have disjoint supports. We define for any N positive integer

$$u_N := \textstyle\sum_{i=1}^N u_i$$

and we test the functional Q_g on u_N. Thanks to the fact that the elements of the sum have disjoint supports and recalling that γ_i are isometries, it is easy to see that the norm in $L^{\frac{2n}{n-2}}(M)$ of u_N equals $N^{\frac{n-2}{2n}}||u||_{\frac{2n}{n-2}}$. For the same reasons, for the numerator we get $E(u_N) = NE(u)$. Therefore:

$$Q_g(u_N) = \frac{E(u_N)}{||u_N||^2_{\frac{2n}{n-2}}} = N^{\frac{2}{n}} Q_g(u) \leq -N^{\frac{2}{n}}\varepsilon$$

The right hand side term goes to $-\infty$ as N tends to infinity: this shows that if the Yamabe constant of an almost homogeneous manifold is negative, it can not be bounded by below. □

As a consequence, to study the existence of a Yamabe minimizer on a almost homogeneous manifolds makes sense only when the Yamabe constant is non-negative. Our goal is to give a new proof of the following:

Theorem 4.20 (Theorem 13 in [Gro13]). *Let (M^n, g) be an almost homogeneous manifold of dimension $n \geq 3$. Assume that the scalar curvature of g is strictly positive and that the Yamabe constant of M is non-negative and strictly smaller that Y_n:*

$$0 \leq Y(M, [g]) < Y_n$$

Then there exists $u \in W^{1,2}(M) \cap (M)$ which solves the Yamabe equation.

Remark 4.21. If the Yamabe constant of (M, g) is non-negative, we can always assume without loss of generality (up to a conformal change) that the scalar curvature of g is non-negative. In most of the steps of our proof we only need $S_g \geq 0$. We will specify later where the positivity of S_g intervenes.

The idea of the proof consists in choosing an appropriate minimizing sequence $\{u_L\}_{L\in\mathbb{N}}$ in $W^{1,2}(M)$, such that the Yamabe quotient $Q_g(u_L)$ converges to $Y(M, [g])$. Therefore, $\{u_L\}_{\in\mathbb{N}}$ is bounded in $W^{1,2}(M)$ and converges to some limit u weakly in $W^{1,2}(M)$, strongly in $L^2(M)$. In order to show that u is not the null limit, we need to find a uniform bound on the norm in $L^\infty(M)$ of the sequence $\{u_L\}_{\in\mathbb{N}}$: this is done by using the Moser iteration technique of Theorem 4.18.

We first define the appropriate minimizing sequence. Let $x_0 \in K$ and consider ball of radius $L \in \mathbb{N}$ centred in x_0. By using the assumption that $Y(M, [g])$ is strictly smaller than Y_n, we can apply the standard argument as in Theorem 2.1 of [SY94], Chapter 5, in order to deduce the following (see also Lemma 5.3 in [AB03]):

Lemma 4.22. *Let (M^n, g) be a almost homogeneous manifold, $n \geq 3$, whose Yamabe constant satisfies $0 \leq Y(M, [g]) < Y_n$. Let $x_0 \in K$ and denote for any $L \in \mathbb{N}$:*

$$Q_L := \inf_{\substack{u \in W_0^{1,2}(B(x_0,L)),\\ u \neq 0}} Q_g(u).$$

Then there exists a function $u_L \in W_0^{1,2}(B(x_0, L))$ solving the Yamabe equation on the ball $B(x_0, L)$ with Dirichlet condition at the boundary:

$$\begin{cases} \Delta_g u_L - a_n S_g u_L = Q_L u_L^{\frac{n+2}{n-2}} \\ u > 0 \text{ in } B(x_0, L), u = 0 \text{ on } \partial B(x_0, L) \\ ||u||_{\frac{2n}{n-2}} = 1. \end{cases}$$

Furthermore, Q_L tends to $Y(M, [g])$ as ℓ goes to infinity.

We can assume that each u_L attains its maximum in a point p_L of K. In fact, if Q_L is such that

$$u_L(Q_L) = \max_{x\in M} u_L(x)$$

there always exist a point $p_L \in K$ and an isometry γ_L which sends p_L to Q_L, $\gamma_L(p_L) = Q_L$. Then we can simply replace u_L with $u_L \circ \gamma_L$, which has still Yamabe quotient equal to Q_L, it is supported in $B(\gamma_L(x_0), L)$ and solves the Yamabe equation with Dirichlet boundary condition on $B(\gamma_L(x_0), L)$

We are in position to prove that the minimizing sequence $\{u_L\}_{L\in\mathbb{N}}$ is uniformly bounded.

Proposition 4.23. *Let (M^n, g) be an almost homogeneous manifold of dimension $n \geq 3$. Assume that the Yamabe constant of M satisfies*

$$0 \leq Y(M, [g]) < Y_n$$

Then the minimizing sequence $\{u_L\}_{L\in\mathbb{N}}$ defined above is uniformly bounded in $L^\infty(M)$.

Proof. By definition of u_L the following weak inequality holds on all of M:

$$\Delta_g u_L \leq (Q_L u_L^{\frac{4}{n-2}} - a_n S_g) u_L \quad \text{on } M.$$

and since in particular S_g is non-negative, we have

$$\Delta_g u_L \leq (Q_L u_L^{\frac{4}{n-2}}) u_L. \tag{4.24}$$

We would like to apply Theorem 4.18 with the potential

$$V = Q_L u_L^{\frac{4}{n-2}}.$$

Remark that, by Sobolev embedding the function $u_L^{\frac{4}{n-2}}$ belongs to $L^{\frac{n}{2}}(M)$, but we need more regularity, since V must be locally integrable to a power q strictly greater than $n/2$. Besides, we need a uniform bound like the one in the assumption (4.20). We are going to reproduce the first step of Moser iteration technique, in order to increase the exponent $n/2$ and be able to apply Theorem 4.18.

Let $\varepsilon > 0$ and let δ be the radius corresponding to $\varepsilon/2$ in inequality (4.17). Choose r and R, $0 < r < R < \delta$ and consider a cut-off function φ supported in $B(x, R)$, equal to one in $B(x, r)$ and such that $|d\varphi| < \frac{2}{R-r}$ on $B(x, R) \setminus B(x, r)$.
Let $\alpha > 1$ and consider φu_L^α. Then by integrations by parts formula and Hölder's inequality we get:

$$\begin{aligned}
\int_M |d(\varphi u_L^\alpha)|^2 dVol_g &\leq \int_M (|d\varphi|^2 u_L^{2\alpha} + \alpha Q_L u_L^{\frac{4}{n-2}} \alpha u_L^{2\alpha}) dVol_g \\
&\leq \int_{B(x,R)} |d\varphi|^2 u_L^{2\alpha} + \alpha Q_L \Big(\int_{B(x,R)} u_L^{\frac{2n}{n-2}} \Big)^{\frac{2}{n}} \Big(\int_{B(x,R)} u_L^{\frac{2\alpha n}{n-2}} \Big)^{\frac{n-2}{n}} \\
&\leq \int_{B(x,R)} |d\varphi|^2 u_L^{2\alpha} + \alpha Q_L \Big(\int_{B(x,R)} u_L^{\frac{2\alpha n}{n-2}} \Big)^{\frac{n-2}{n}}.
\end{aligned}$$

where we used that u_L has norm equal to one in $L^{\frac{2n}{n-2}}(M)$. By applying Lemma 4.17 to φu_L^α we obtain:

$$\Big(Y_n - \frac{\varepsilon}{2}\Big)\Big(\int_M (\varphi u^\alpha)^{\frac{2n}{n-2}}\Big)^{\frac{n-2}{n}} \leq \int_{B(x,R)} |d\varphi|^2 u_L^{2\alpha} + \alpha Q_L \Big(\int_{B(x,R)} u_L^{\frac{2\alpha n}{n-2}}\Big)^{\frac{n-2}{n}}$$

and as a consequence

$$(Y_n - \varepsilon - \alpha Q_L)\Big(\int_{B(x,R)} u_L^{\frac{2\alpha n}{n-2}}\Big)^{\frac{n-2}{n}} \leq \frac{2}{(R-r)^2}\int_{B(x,R)} u_L^{2\alpha}$$

Recall that Q_L tends to $Y(M,[g]) < Y_n$. Then there exists L_0 such that for any $L > L_0$, we have $Q_L < Y_n - 2\varepsilon$. Now we can choose $\alpha > 1$ in such a way that

$$\alpha Q_L < Y_n - \frac{3}{2}\varepsilon$$

We then have a positive constant on the left-hand side of the previous inequality.

Beside we can choose α to be between 1 and $\frac{n}{n-2}$, and ε, R consequently, so that $L^{2\alpha}(M)$ is continuously embedded in $W^{1,2}(M)$ and the right-hand side is bounded by the norm of u_L in $W^{1,2}(M)$. Therefore we get:

$$||u_L||_{L^{\frac{2\alpha n}{n-2}}(B(x,r))} \leq B||u_L||_{W^{1,2}(M)}$$

Now u_L is a minimizing sequence for the Yamabe functional, so it is a bounded sequence in $W^{1,2}(M)$. This means that we have obtained a bound on the norm of u_L in $L^{\frac{2\alpha n}{n-2}}(B(x,r))$ which is independent of L, x and r. Observe that such bound easily implies that u_L belongs to $L_{loc}^{\frac{2\alpha n}{n-2}}(M)$, since we can cover any compact by a finite number of balls of radius r.

In particular we obtained that u_L belongs to $L_{loc}^q(M)$ for $q = \alpha n/2 > n/2$, and moreover the assumption (4.20) holds for the power $4/(n-2)$ of u_L. This allows us to apply the Theorem 4.18. Therefore we obtain that u_L belongs to $L^\infty(M)$ and the sequence is uniformly bounded in $L^\infty(M)$, since any of the constants depends on L. $\square$

Observe that in the previous Proposition we do not need to assume that the scalar curvature is positive. This is in turn fundamental to prove Theorem 4.20, as the following argument shows:

Proof of Theorem 4.20. The minimizing sequence u_L is bounded in $W^{1,2}(M)$, so there exists a function u in $W^{1,2}(M)$ such that, up to a subsequence, u_L converges to u weakly in $W^{1,2}(M)$ and strongly in $L^p(M)$ for any $p \in [1, \frac{2n}{n-2})$.

Moreover, we have shown that $\{u_L\}_L$ is uniformly bounded in $L^\infty(M)$, then by regularity theorem (see for example [LP87], Theorem 4.1) we deduce that up to a subsequence u_L converges to u in $C_{loc}^2(M)$ and that u belongs to $C^2(M)$. We have to check that u is not the null limit. Consider $p_L \in K$ such that

$$u_L(p_L) = \sup_M u_L = M_L.$$

Then we have $\Delta u_L(p_L) \geq 0$ and since u_L solves the Yamabe equation we get

$$S_g(p_L)M_L \geq Q_L M_L^{\frac{n+2}{n-2}} \Rightarrow M_L^{\frac{4}{n-2}} \geq \frac{S_g(p_L)}{Y_n} > 0$$

Where we are using here that by assumption S_g is strictly positive on M, not only non-negative. Since p_L belongs to a compact K for any L, $S_g(p_L)$ cannot tend to zero as L goes to infinity; in particular, M_L is bounded by below by a positive constant and the uniform limit u cannot vanish.
Besides, the convergence in $C^2_{loc}(M)$ implies the strong convergence in $L^{\frac{2n}{n-2}}_{loc}(M)$: hence, for any $\varphi \in C^\infty_0(M)$ we can pass to the limit in

$$\int_M ((du_L, d\varphi)_g + a_n S_g u_L \varphi) dv_g = Q_L \int_M \varphi u_L^{\frac{n+2}{n-2}} dv_g.$$

and obtain that $u \in W^{1,2}(M) \cap L^\infty(M)$ is a weak, then strong, solution of the Yamabe equation. □

4.3.2 Application to the local Yamabe constant

One of the interests of the previous result is that it can be applied to a product between the hyperbolic space $\mathbb{H}^{n-d}$ and a compact smooth manifold Z^d, and we know from the previous chapter that such product is conformal to the cone $C(S)$ over the j-fold spherical suspension of Z. Furthermore, if Z is an Einstein manifold, than $C(S)$ carries an Einstein metric with scalar curvature equal to $n(n-1)$, conformal to the exact cone metric: therefore, if we are able to find a Yamabe minimizer on $\mathbb{H}^{n-d} \times Z$, there exists a Yamabe minimizer on $C(S)$ and we can apply Corollary 4.15 in order to compute the value of the Yamabe constant.

We have the following:

Lemma 4.24. *Let (Z^d, k) be a compact Einstein manifold of dimension $d > \frac{n}{2}$ such that $Ric_k = (d-1)k$. Consider the Riemannian product $X = \mathbb{H}^{n-d} \times Z^d$ endowed with the product metric $g = g_{\mathbb{H}^{n-d}} + k$. If (X, g) has non-negative Yamabe constant, one of the following possibilities holds:*

(i) *either $Y(X, [g]) = Y_n$;*

(ii) *or $Y(X, [g]) < Y_n$ and there exists a Yamabe minimizer.*

The condition $d > \frac{n}{2}$ ensures that the scalar curvature of g is strictly positive, and that we can apply Theorem 4.20. This result allows us to find the local Yamabe constant of a stratified space with simple edges carrying an Einstein metric, provided that they have dimension $(n-d-1)$ with d greater than $n/2$. This is less general but clearly agrees with the previous result in Proposition 4.11 holding for any link endowed with an Einstein metric.

Proposition 4.25. *Let (X, g) be a stratified space of dimension n, with one stratum of dimension $n-d-1$. Assume that its link is a compact smooth manifold Z^d with $d > \frac{n}{2}$ endowed with an Einstein metric k such that $Ric_k = (d-1)k$. Then the local Yamabe constant of X is equal to:*

$$Y_\ell(X) = \min\left\{Y_n, \left(\frac{\mathrm{Vol}_k(Z)}{\mathrm{Vol}(\mathbb{S}^d)}\right)^{\frac{2}{n}} Y_n\right\}.$$

Proof. Proposition 1.4(b) in [ACM14] ensures that the local Yamabe constant is positive, then in this case we have $Y(\mathbb{H}^{n-d} \times Z^d, [g_{\mathbb{H}^{n-d}} + k]) > 0$ and we can apply the previous Lemma in order to deduce that either the Yamabe constant of $\mathbb{H}^{n-d} \times Z^d$ is equal to Y_n, or there exists a Yamabe minimizer. In this case, the cone $C(S)$ endowed with the Einstein metric $g_c = dt^2 + \cos^2(t)h$ admits a Yamabe minimizer, and then we can apply Corollary 4.15: the metric g_c attains the Yamabe constant of $C(S)$, which is equal to:

$$Y(C(S), [g_c]) = \frac{n(n-2)}{4} \mathrm{Vol}_{g_c}(C(S))^{\frac{n}{2}}.$$

Since the Yamabe constant of the sphere $\mathbb{S}^n$ is:

$$Y_n = \frac{n(n-2)}{4} \mathrm{Vol}(\mathbb{S}^n)^{\frac{n}{2}}.$$

we can reformulate the previous and get:

$$Y(C(S), [g_c]) = Y_n \left(\frac{\mathrm{Vol}_{g_c}(C(S))}{\mathrm{Vol}(\mathbb{S}^n)}\right)^{\frac{n}{2}}$$

This leads exactly to the same result as in Lemma 4.3. □

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UNIVERSITÉ DE NANTES

Thèse de Doctorat

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Le Problème de Yamabe sur les espaces stratifiés.

The Yamabe Problem on stratified spaces.

Résumé
On étudie une classe d'espaces métriques singuliers, les espaces stratifiés, et on se propose d'étendre à ces derniers des résultats de géométrie riemannienne et d'analyse sur les variétés. Dans une première partie, on montre l'existence d'une borne inférieure pour le bas du spectre du Laplacien, sous une hypothèse géométrique de minoration de la courbure de Ricci. Cela permet également de démontrer l'existence d'une inégalité de Sobolev dont les constantes dépendent uniquement du volume et de la dimension de l'espace, et d'une borne supérieure pour le diamètre. En outre, on prouve que la borne pour le diamètre est atteinte si et seulement si celle pour le bas du spectre l'est aussi. La deuxième partie de ce manuscrit est dédiée aux conséquences des résultats précédents sur le problème de Yamabe pour un espace stratifié : ce problème consiste à chercher une métrique conforme à courbure scalaire constante, et l'existence d'une solution dépend d'un invariant conforme, la constante de Yamabe locale, dont la valeur est en général inconnue. On montre que celle-ci peut-être calculée en un grand nombre de cas, lorsque une hypothèse géométrique sur le lieu singulier est vérifiée. On utilise des techniques liées aux inégalités isopérimétrique et de Sobolev. Enfin, on donne une classe d'exemples pour lesquels on peut prouver qu'une métrique conforme à courbure scalaire constante existe.

Mots clés
Espaces stratifiés, bas du spectre, inégalité de Sobolev, théorème de Myers, problème de Yamabe, constante de Yamabe locale, inégalité isopérimétrique.

Abstract
We study a class of singular metric spaces, stratified spaces, with an approach whose goal is to extend to these latter some tools and results of Riemannian geometry and analysis on smooth manifolds. In a first part, we show the existence of a lower bound for the bottom of the spectrum of the Laplacian, under the assumption that the Ricci curvature is bounded by below. This allows us to prove also the existence of a Sobolev inequality whose constants only depend on the volume and of the dimension of the space, and of an upper bound for the diameter. Furthermore, we prove that the bound for the diameter is attained if and only if the one for the bottom of the spectrum is attained as well.
The second part is devoted to the direct consequences of the previous results on the Yamabe problem on a stratified space: this problem consists in looking for a conformal metric with constant scalar curvature, and the existence of a solution depends on a conformal invariant, the local Yamabe constant, whose value is generally unknown. We show that this latter can be computed in a large number of cases, when a geometric hypothesis on the singular set is verified. We use techniques which are related to the Sobolev and the isoperimetric inequalities. Finally, we give a class of examples for which we can prove the existence of a conformal metric with constant scalar curvature.

Key Words
Stratified spaces, bottom of the spectrum, Sobolev inequality, Myers theorem, Yamabe problem, local Yamabe constant, isoperimetric inequality.